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A degree-theoretic approach to geometric equations on manifolds with symmetries

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Abstract

This thesis presents new solutions to three geometric partial differential equations: the prescribed Ricci curvature equation, the Einstein equation and the quasi-Einstein equation. We find solutions of the prescribed Ricci curvature problem in the homogeneous setting, in which the problem reduces to a system of algebraic equations. We find new solutions of the Einstein and quasi-Einstein equations on cohomogeneity one manifolds with boundary, in which case the equations reduce to systems of ordinary differential equations.

The new solutions of the three geometric partial differential equations are produced by using *topological degree theory*. This theory assigns a degree to an equation which provides a weighted count of the solutions we expect the equation to have inside a set using only data at the boundary of the set. A key property of the degree is that it is homotopy invariant, meaning that the degree is preserved by any continuous deformation of the equation, as long as solutions to these deformed equations do not occur on the boundary. This property allows one to analyse existence of solutions of an equation by continuously deforming it into another simpler equation and examining the new equation instead.

In Chapter 1, some history of solving the three equations is discussed, and our results are stated informally. In Chapter 2, the homogeneous and cohomogeneity one assumptions are discussed in more detail, and the main results of this thesis are stated precisely. In Chapter 3, we briefly describe the construction of the Brouwer degree, which concerns the solvability of algebraic equations, and state its useful properties. We then use the theory to produce solutions of the prescribed Ricci curvature problem. In Chapter 4, we briefly describe the construction of the Schauder degree, which concerns the solvability of differential equations, and state its useful properties. Deformations for the Einstein and quasi-Einstein equations into simpler equations are also introduced. These simpler equations are studied in Chapter 5, allowing us to find new solutions of the Einstein and quasi-Einstein equations.

Declaration by author

This thesis is composed of my original work, and contains no material previously published or written by another person except where due reference has been made in the text. I have clearly stated the contribution by others to jointly-authored works that I have included in my thesis.

I have clearly stated the contribution of others to my thesis as a whole, including statistical assistance, survey design, data analysis, significant technical procedures, professional editorial advice, financial support and any other original research work used or reported in my thesis. The content of my thesis is the result of work I have carried out since the commencement of my higher degree by research candidature and does not include a substantial part of work that has been submitted to qualify for the award of any other degree or diploma in any university or other tertiary institution. I have clearly stated which parts of my thesis, if any, have been submitted to qualify for another award.

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Publications included in this thesis

1. [16] **T. Buttsworth**, The Dirichlet problem for Einstein metrics on cohomogeneity one manifolds, *Ann. Glob. Anal. Geom*, 54, 1, 2018.

2. [17] **T. Buttsworth**, Cohomogeneity-one quasi-Einstein metrics, *J. Math. Anal. Appl.*, 470, 1, 2019.

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No animal or human subjects were involved in this research.

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Contents

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Chapter 1

Introduction

Equations of fundamental interest in Riemannian geometry include the prescribed Ricci curvature equation, the Einstein equation and the quasi-Einstein equation. In this thesis, we prove existence of solutions to these equations in a number of interesting cases using topological degree theory. In the introduction, we will discuss these equations and informally state our existence results. For this introduction, M will denote a smooth manifold, g a Riemannian metric on M, and $Ric(g)$ the Ricci curvature of g.

1.1 The prescribed Ricci curvature equation

The prescribed Ricci curvature problem involves finding a Riemannian metric g that solves the prescribed Ricci curvature equation

$$Ric(g) = T \tag{1.1}$$

for a given tensor field T on M. The local solvability of (1.1) is relatively well understood. Indeed, DeTurck [26] demonstrates that if T is non-degenerate at a point $p \in M$, then there exists a Riemannian metric g such that (1.1) holds on some neighbourhood of p.

Theorem 1.1.1. *Choose a smooth manifold M of dimension at least three, and choose a point $p \in M$. If T is a smooth symmetric $(0,2)$-tensor field defined in a neighbourhood of p such that $T(p)$ is non-degenerate, then there exists a smooth Riemannian metric g satisfying* (1.1) *in a neighbourhood of p.*

DeTurck and Goldschmidt [28] also show that (1.1) holds in a neighbourhood of a point $p \in M$ if T has constant rank near p and satisfies certain other constraints.

One would like to know when it is possible to solve (1.1) on all of M, not just in some neighbourhood. Even if T is non-degenerate on all of M, there can be obstructions to existence. DeTurck and Koiso [29] show that if M is closed and T is positive-definite, then there is a constant $c > 0$ such that cT does not coincide with $Ric(g)$ for *any* Riemannian metric g (see Chapter 5 of [10] for a more detailed discussion of this result). This obstruction to global existence is related to the scaling invariance of the Ricci curvature, and so it appears that a more reasonable problem is to find a Riemannian metric g and a constant $c > 0$ such that

$$Ric(g) = cT. \tag{1.2}$$

Hamilton [34] and DeTurck [27] have examined this problem and use two different versions of the inverse function theorem to prove existence of solutions to (1.2) for T close to various Einstein metrics. Indeed, in [34], Hamilton proves the following with the Nash-Moser theorem:

Theorem 1.1.2. *Let g_0 be a metric on the sphere $\mathbb{S}^n$ of constant curvature 1 so $Ric(g_0) = g_0$. For every T near g_0, there exists a unique constant c so that (1.2) holds for some g near g_0, and g is also the unique solution in the neighbourhood of g_0 if g is required to have the same volume as g_0.*

More generally, DeTurck in [27] obtains the following:

Theorem 1.1.3. *Let g_0 be an Einstein metric on a compact manifold M with $Ric(g_0) = g_0$ for which the kernel of the Lichnerwowicz Laplacian is one-dimensional. Fix p greater than the dimension of M. Then there exists an $\varepsilon > 0$ so that if T is a smooth Riemannian metric satisfying $|T - g_0|_{L^p(M,g_0)} < \varepsilon$, then there exists a positive contstant $c > 0$ and a smooth Riemannian metric g satisfying (1.2).*

There are several other results on the global existence problem for (1.1) and (1.2), but many are perturbative in nature and rely heavily on the inverse function theorem (see, e.g., [24, 25]).

Finding general results about the solvability of (1.2) tends to be difficult, so an effort has been made to study the problem in simpler settings. For example, when our manifold M is acted on transitively by some Lie group G, and we require that g and T are invariant under the action of G, (1.2) becomes a system of algebraic equations. This problem has been studied in [32, 34, 44, 46], for example. In [34], Hamilton studies the case that G is $SO(3)$ and in [44], Pulemotov studies the case that M is a homogeneous space G/H with H a maximal connected Lie subgroup of G:

Theorem 1.1.4. *Suppose H is a maximal connected Lie subgroup of a compact and connected Lie group G. Let T be a G-invariant Riemannian metric on the homogeneous space G/H. Then there exists a G-invariant Riemannian metric g and a positive constant c solving (1.2).*

The case that H is not maximal connected in G is treated by Gould and Pulemotov in [32], and by Pulemotov in [46].

After the homogeneous setting, the next natural step is requiring that our d-dimensional manifold M has a G-action whose principal orbits are $(d-1)$-dimensional. Here, we say our manifold is cohomogeneity one, and it is natural to restrict attention to those T and g that are G-invariant. In this case, (1.2), and many other geometric equations, become systems of ODEs rather than systems of PDEs, and results about existence are easier to obtain. The cohomogeneity one problem for (1.1) and (1.2) has been studied in [19, 34, 43].

In this thesis, we examine (1.2) in the case that M is the homogeneous space $\mathbb{S}^{4n+3} = Sp(n+1)/Sp(n)$, and T is assumed to be positive-definite and $Sp(n+1)$-invariant. Using Brouwer degree theory, we obtain an existence result for this problem. Our techniques are not perturbative. Moreover, the result cannot be recovered by using the inverse function theorem.

1.2 The Einstein equation

Einstein's theory of relativity suggests that in the presence of matter described by a tensor field T, the geometry of space-time should be determined by a Lorentzian metric g satisfying the celebrated Einstein field equations

$$Ric(g) - \frac{S(g)g}{2} = T + \mu g, \tag{1.3}$$

where μ is referred to as the 'cosmological constant.' If space-time is assumed to have no matter, then $T = 0$, and (1.3) becomes the Einstein equation

$$Ric(g) = \lambda g, \tag{1.4}$$

where $\lambda \in \mathbb{R}$ is constant. Lorentzian and Riemannian metrics g satisfying (1.4) are referred to as Einstein metrics with Einstein constant λ. Riemannian metrics solving (1.4) are of fundamental interest in geometry because they can be viewed as having constant curvature; see the introduction to the subject in Chapter 0 of [10].

For open and closed manifolds, there are several results relating to the solvability of (1.4), and a large and detailed survey of some classical results appears in [10]. Some more recent results are available in [7]. In addition to open and closed manifolds, it is natural to consider the Einstein equation, as well as other geometric PDEs, on manifolds with boundary, in which case one prescribes various boundary conditions. Anderson studies the problem of solving (1.4) on manifolds with boundary in [3], and considers Dirichlet conditions, Neumann conditions, and prescribing the conformal class of the metric (prescribing the metric up to multiplication by a scalar function) as well as the mean curvature at the boundary. Anderson demonstrates that on certain Hölder spaces, the linearisation of the Einstein equation with Dirichlet conditions has an infinite-dimensional kernel. This property implies that the problem is not Fredholm, which makes it difficult to study in general. However, this equation is Fredholm under the prescription of mean curvature and conformal class.

The issues of choosing appropriate boundary conditions have also come up in relation to other geometric equations. For example, in the study of the Ricci flow, Pulemotov and Gianniotis study boundary conditions involving the mean curvature and conformal class in [42] and [31], respectively, while Shen and Pulemotov study Robin-type and Neumann-type boundary conditions in [47] and [45], respectively. Pulemotov also considers Dirichlet boundary conditions for the prescribed Ricci curvature problem in [43].

As with the prescribed Ricci curvature equation, the Einstein equation (1.4) has been studied extensively in the case that M is acted on transitively by a Lie group G and g is G-invariant. Solutions of this simpler problem are known as homogeneous Einstein metrics and have been studied in, for example, [4, 14, 15, 36, 41, 49]; see also the surveys [6] and [48]. Einstein metrics have also been studied in the cohomogeneity one setting. One of the first examples of a cohomogeneity one Einstein metric appeared in [39], and subsequently, the general theory began to be developed in the 1980s by Bérard-Bergery in [9], and by Page and Pope in [40]. Since then, there has been a large amount of work done in the area of cohomogeneity one Einstein metrics. For example, in [12, 13], Böhm examines the Einstein equation on certain cohomogeneity one manifolds, and obtains existence and non-existence results. In [30], Eschenburg and Wang study the initial-value problem for the Einstein metric in a neighbourhood of a fixed orbit. The Einstein equation has also been studied by Dancer and Wang in [22] by viewing the ODE as a Hamiltonian flow. The cohomogeneity one setting was also used in the study of Ricci solitons by Dancer and Wang in [23] and in the work on the Ricci flow done by Pulemotov in [45] and by Bettiol and Krishnan in [11].

In this thesis, we study the Einstein equation on cohomogeneity one manifolds M subject to boundary conditions. We assume that M is $G/H \times [0,1]$, where G/H is a compact homogeneous space. The boundary of our manifold M is $(G/H \times \{0\}) \cup (G/H \times \{1\})$. The Dirichlet problem in this case consists in finding Einstein metrics that coincide with two fixed G-invariant Riemannian metrics $\hat{g}_0$ and $\hat{g}_1$ on $G/H \times \{0\}$ and $G/H \times \{1\}$, respectively. Using Schauder degree theory, we demonstrate that we can always find a one-parameter family of Einstein metrics after we impose the "monotypic" hypothesis on the compact homogeneous space G/H.

1.3 The quasi-Einstein equation

Choose some non-negative real number m, a smooth function $u : M \to \mathbb{R}$ and a Riemannian metric g. The m–Bakry–Emery tensor $Ric_u^m(g)$ is defined by

$$Ric_u^m(g) = Ric(g) + Hess(u) - m\, du \otimes du,$$

where $Hess(u)$ denotes the Hessian of the function u with respect to g. The m–Bakry–Emery tensor can be thought of as an extension of the Ricci curvature because the two coincide when the function u is constant. When $m = 0$, the m–Bakry–Emery tensor coincides with the usual Bakry–Emery tensor appearing in [8]. The case that m is strictly positive has been studied, for example, in [20, 21, 37, 50]. One setting in which this case arises is the study of the smooth metric measure space $(M, g, e^{-u} dV(g))$, where $dV(g)$ is the volume form of the Riemannian metric g; for example, see [50].

Motivated by the extensive theory of Einstein metrics, one would like to develop a theory for solutions of

$$Ric_u^m(g) = \lambda g. \tag{1.5}$$

In particular, one would like to know under what circumstances solutions exist, and how they behave. A pair (g, u) solving (1.5) is called a *quasi-Einstein metric*. Clearly, a quasi-Einstein metric is an Einstein metric if u is constant, but there is a relationship between the two concepts even if u is non-constant. Indeed, as discussed in [37], Einstein metrics on warped product spaces arise as solutions to the quasi-Einstein equation (1.5). This observation was used by Case in [20] to demonstrate non-existence of Einstein metrics on warped product spaces under certain conditions. The relationship between Einstein and quasi-Einstein metrics also appears in [21], where the authors prove rigidity results for (1.5).

We examine (1.5) on the cohomogeneity one manifold $G/H \times [0,1]$ subject to G-invariant Dirichlet conditions, where G/H is a homogeneous space satisfying the "monotypic" hypothesis. In this case, the Dirichlet problem consists in finding a quasi-Einstein metric (g,u) on $G/H \times [0,1]$ so that for $i = 0,1$, (g,u) coincides with $(\hat{g}_i, u(i))$ when restricted to $G/H \times \{i\}$, where $\hat{g}_i$ is a fixed G-invariant Riemannian metric on G/H, and $u(i)$ is a fixed real number. Despite the relationship between quasi-Einstein metrics and Einstein metrics on warped product spaces, being able to solve the Dirichlet problem for Einstein metrics does not immediately imply the solvability of the Dirichlet problem for quasi-Einstein metrics. In particular, the study of the Dirichlet problem for cohomogeneity one Einstein metrics is not immediately helpful in the study of the Dirichlet problem for cohomogeneity one quasi-Einstein metrics. However, we are still able to use Schauder degree theory to prove existence of a one-parameter family of solutions to the Dirichlet problem for (1.5).

Chapter 2

Symmetry reductions and statement of results

In this chapter, we examine the homogeneity and cohomogeneity one assumptions in more detail. Doing so will also allow us to state the main results of this thesis.

2.1 The homogeneous prescribed Ricci curvature equation on $\mathbb{S}^{4n+3}$

To state our result on the prescribed Ricci curvature equation, we need the notion of a homogeneous space.

Definition 2.1.1. *The manifold M is said to be a homogeneous space if there is a Lie group G acting transitively and smoothly on M.*

If M is a homogeneous space with Lie group action G, then M can be identified with the coset space G/H, where H is the closed isotropy subgroup. If we require that both T and g are homogeneous, i.e., they are G-invariant, then (1.2) becomes a system of algebraic equations. In order to study the prescribed Ricci curvature equation for homogeneous g and T, we need to understand what $Ric(g)$ looks like. It is helpful to examine the Ricci curvature of a G-invariant Riemannian metric at the level of Lie algebras. Indeed, we denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively. By assuming that H is compact, we can take a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is an $Ad_G(H)$-invariant subspace of $\mathfrak{g}$. In this way, a G-invariant Riemannian metric corresponds to an $Ad_G(H)$-invariant inner product $(\cdot,\cdot)$ on $\mathfrak{p}$. In fact, Proposition 5.1 of [5] implies that this correspondence is one-to-one.

If g is a G-invariant Riemannian metric on the homogeneous space G/H, the diffeomorphism invariance of the Ricci curvature implies that $Ric(g)$ is also G-invariant, and so can also be interpreted as an $Ad_G(H)$-invariant bilinear form on $\mathfrak{p}$. The following theorem gives an explicit description of this bilinear form.

Theorem 2.1.2 (Corollary 7.33 in [10])**.** *As a symmetric bilinear form on* $\mathfrak{p}$*, the Ricci curvature is given by*

$$Ric(g)(X,X) = -\frac{1}{2}\sum_i |[X,X_i]_{\mathfrak{p}}|^2 - \frac{1}{2}B(X,X) + \frac{1}{4}\sum_{i,j}([X_i,X_j]_{\mathfrak{p}},X)^2 - ([Z,X]_{\mathfrak{p}},X).$$

Here, $B(X,Y) = tr(ad(X)\circ ad(Y))$ *is the Killing form of* $\mathfrak{g}$*,* $\{X_i\}$ *is an orthonormal basis for* $\mathfrak{p}$ *with respect to* $g = (\cdot,\cdot)$*, and* $Z \in \mathfrak{p}$ *is chosen to be the unique element so that* $(Z,X) = tr(ad(X))$ *for each* $X \in \mathfrak{p}$*.*

In this thesis, we focus on the case that M is the sphere $\mathbb{S}^{4n+3}$ for some $n \in \mathbb{N}$. Any sphere can be given a homogeneous space structure through the transitive action of the special orthogonal group, but $\mathbb{S}^{4n+3}$ admits other homogeneous structures; see, e.g. Example 6.16 of [1]. For instance, we can give $\mathbb{S}^{4n+3}$ a homogeneous structure by embedding it inside $\mathbb{R}^{4(n+1)}$, which we identify with $\mathbb{H}^{n+1}$, the space of $(n+1)$-tuples of quaternions. The action of the quaternionic unitary group $Sp(n+1)$ leaves $\mathbb{S}^{4n+3}$ invariant with isotropy subgroup $Sp(n)$. Consequently, $\mathbb{S}^{4n+3}$ appears as the homogeneous space $Sp(n+1)/Sp(n)$.

To examine $Sp(n+1)$-invariant metrics on $\mathbb{S}^{4n+3}$, we let $\mathfrak{g}$ be the Lie algebra of $G = Sp(n+1)$, $\mathfrak{h}$ the Lie algebra of $H = Sp(n)$, and $\mathfrak{p}$ an $Ad_G(H)$-invariant subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{p}$. We can find a decomposition $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$, where $\mathfrak{p}_0$ can be identified with $\mathfrak{sp}(1)$ (the Lie algebra of $Sp(1)$), and $\mathfrak{p}_1$ can be identified with the Lie algebra of the quarternionic group $\mathbb{H}^n$. We equip $\mathfrak{p}_1$ with the standard inner product $\hat{g}$. Choose a basis $\{X_1, X_2, X_3\}$ of $\mathfrak{p}_0$ satisfying

$$[X_i,X_j] = \sum_{k=1}^{3} 2\varepsilon_{ijk}X_k, \tag{2.1}$$

where ε_{ijk} is the Levi-Civita symbol. Then up to isometry and scaling, every $Sp(n+1)$-invariant metric g on $\mathbb{S}^{4n+3}$ has the form

$$g|_{\mathfrak{p}_1} = \hat{g}, \qquad g(\mathfrak{p}_0,\mathfrak{p}_1) = 0, \qquad g(X_i,X_j) = x_i\delta_{ij}, \tag{2.2}$$

where x_i are some positive numbers (see [51] for more details). Using Theorem 2.1.2, we see that the Ricci curvature of such a metric is given by

$$\begin{aligned} Ric(g)|_{\mathfrak{p}_1} &= (4n+8-2(x_1+x_2+x_3))\hat{g}, \\ Ric(g)(\mathfrak{p}_0,\mathfrak{p}_1) &= 0, \\ Ric(g)(X_i,X_j) &= 0 \text{ for } i \neq j, \\ Ric(g)(X_i,X_i) &= 4nx_i^2 + 2\frac{(x_i^2-(x_j-x_k)^2)}{x_jx_k}, \end{aligned}$$

where the j and k in the last line are chosen so that i,j,k are pairwise distinct.

If T itself is a $Sp(n+1)$-invariant Riemannian metric, then we can assume that it has the form

$$T|_{\mathfrak{p}_1} = \hat{g}, \qquad T(\mathfrak{p}_0,\mathfrak{p}_1) = 0, \qquad T(X_i,X_j) = T_i\delta_{ij},$$

for some positive numbers T_1, T_2, T_3. If we search for solutions of (1.2) with g having the form of (2.2) for the same decomposition $\mathfrak{p} = \mathfrak{p}_0 \oplus \mathfrak{p}_1$, then the problem is reduced to solving the following system of equations for positive numbers x_1, x_2, x_3 and c:

$$\begin{aligned} c &= 4n + 8 - 2(x_1 + x_2 + x_3), \\ cT_i &= 4nx_i^2 + 2\frac{(x_i^2 - (x_j - x_k)^2)}{x_j x_k}, \qquad i = 1,2,3. \end{aligned} \tag{2.3}$$

Here, $j,k \in \{1,2,3\}$ are chosen so that i, j, k are pairwise distinct.

Theorem 2.1.3. *Assume that the homogeneous metric T satisfies*

$$\frac{1}{T_i} < 2n+4, \quad i = 1,2,3.$$

Then there exists a homogeneous metric g such that $Ric(g) = cT$ for some $c > 0$.

We prove this result in Chapter 3 by finding a solution of (2.3).

Remark 2.1.4. *We will see in Chapter 3 that the condition $\frac{1}{T_i} < 2n+4$ in Theorem 2.1.3 is not necessary for existence of solutions to* (2.3). *After the proof of Theorem 2.1.3, we discuss another condition that may be both necessary and sufficient for existence of solutions to* (2.3). *We do not discuss this alternate condition here because it is difficult to state.*

2.2 The Dirichlet problem for Einstein and quasi-Einstein metrics on cohomogeneity one manifolds

To state our main results for the Einstein equation and the quasi-Einstein equation, we need the notion of a cohomogeneity one manifold.

Definition 2.2.1. *Let G be a Lie group acting smoothly on a smooth manifold M. If G has an orbit of co-dimension one, we call M a cohomogeneity one manifold.*

If M is a cohomogeneity one manifold with Lie group G, the quotient M/G is one-dimensional. An example of a cohomogeneity one manifold is the sphere $\mathbb{S}^2$ centered in $\mathbb{R}^3$, equipped with the Lie group $G = \mathbb{S}^1$ acting via rotations around the z-axis. In this example, the orbits of G in M are the lines of latitude, alongside the north and south poles.

Some of the basic theory of cohomogeneity one manifolds is outlined in [30], [33] and [1]. Most of the previous literature focuses on the case where G has non-principal orbits in M; these are often required to compactify a manifold without introducing a boundary. However, in this thesis, we focus on the case that M is a cohomogeneity one manifold $G/H \times [0,1]$, where G/H is a homogeneous space with compact Lie group G. This manifold M has a boundary given by $(G/H \times \{0\}) \cup (G/H \times \{1\})$.

We are using r to parameterise the interval $[0,1]$ in $G/H \times [0,1]$. If we assume that the Riemannian metric g and the function u are G-invariant, then up to a diffeomorphism, g and u have the form

$$\begin{aligned} g &= h^2\, dr \otimes dr + g_r, \\ u &= u(r), \end{aligned} \tag{2.4}$$

where h is some positive constant, $u(r)$ is a real number depending smoothly on r, and g_r is a one-parameter family of homogeneous Riemannian metrics on the homogeneous space G/H.

We denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H respectively. We fix a reference bi-invariant metric Q on G, and let $\mathfrak{p}$ be the Q-orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. We assume that the homogeneous space G/H satisfies the monotypic hypothesis.

Definition 2.2.2. *We say the homogeneous space G/H satisfies the* monotypic hypothesis *if $\mathfrak{p}$ decomposes into Q-orthogonal $Ad_G(H)$-irreducible submodules $\mathfrak{p}_i$ that are pairwise inequivalent.*

The sphere $\mathbb{S}^n = G/H = SO(n+1)/SO(n)$ is an example of a homogeneous space that satisfies the monotypic hypothesis because in this case, $\mathfrak{p}$ is already $Ad_G(H)$-irreducible. For another example, take the sphere $\mathbb{S}^{15}$ which can be given another homogeneous structure with $\mathbb{S}^{15} = Spin(9)/Spin(7)$; this time $\mathfrak{p}$ decomposes into two inequivalent and irreducible submodules.

If G/H satisfies the monotypic hypothesis, then the decomposition $\mathfrak{p} = \bigoplus_{i=1}^n \mathfrak{p}_i$ is unique up to the order of summands, and Schur's Lemma implies that any $Ad_G(H)$-invariant inner product on $\mathfrak{p}$ respects the decomposition $\mathfrak{p} = \bigoplus_{i=1}^n \mathfrak{p}_i$, and coincides with a scalar multiple of Q on each $\mathfrak{p}_i$. Therefore, the G-invariant metric g_r has the form

$$g_r(X,Y) = \sum_{i=1}^n f_i^2(r) Q(pr_{\mathfrak{p}_i}X, pr_{\mathfrak{p}_i}Y)$$

for some smooth functions $f_i : [0,1] \to \mathbb{R}^+ = (0,\infty)$. Here, $pr_{\mathfrak{p}_i}$ means the Q-orthogonal projection onto $\mathfrak{p}_i$. Now, Lemma 3.1 of [43] yields that the Ricci curvature of the Riemannian metric in (2.4) is given by $Ric(g) = H(r)dr \otimes dr + R_r$, where

$$H(r) = -\sum_{k=1}^n d_k \left(\frac{f_k''}{f_k}\right)$$

and

$$R_r(X,Y) = \\ \sum_{i=1}^n \left(\frac{\beta_i}{2} + \sum_{k,l=1}^n \gamma_{ik}^l \frac{f_i^4 - 2f_k^4}{4f_k^2 f_l^2} - \frac{f_i f_i'}{h} \sum_{k=1}^n d_k \frac{f_k'}{h f_k} + \frac{f_i'^2}{h^2} - \frac{f_i f_i''}{h^2} \right) Q(pr_{\mathfrak{p}_i}X, pr_{\mathfrak{p}_i}Y),$$

for $X,Y \in \mathfrak{p}$. Here, β_i are constants which relate Q to the Killing form on G, γ_{ik}^l are the structure constants for the homogeneous space and d_i is the dimension of the submodule $\mathfrak{p}_i$. See [49] or [43] for the precise definitions of these numbers. Therefore the Einstein and quasi-Einstein equations become

$$\begin{aligned} -\sum_{k=1}^n d_k \frac{f_k''}{f_k} &= h^2\lambda, \\ \frac{h^2\beta_i}{2f_i^2} + h^2 \sum_{k,l=1}^n \gamma_{ik}^l \frac{f_i^4 - 2f_k^4}{4f_i^2 f_k^2 f_l^2} - \frac{f_i'}{f_i} \sum_{k=1}^n d_k \frac{f_k'}{f_k} + \frac{f_i'^2}{f_i^2} - \frac{f_i''}{f_i} &= h^2\lambda, \qquad i = 1, \cdots, n, \end{aligned} \tag{2.5}$$

and

$$
\begin{aligned}
u'' - m(u')^2 - \sum_{k=1}^{n} d_k \frac{f_k''}{f_k} &= h^2\lambda, \\
u' \frac{f_i'}{f_i} + \frac{h^2\beta_i}{2f_i^2} + h^2 \sum_{k,l=1}^{n} \gamma_{ik}^{l} \frac{f_i^4 - 2f_k^4}{4f_i^2 f_k^2 f_l^2} - \frac{f_i'}{f_i} \sum_{k=1}^{n} d_k \frac{f_k'}{f_k} + \frac{f_i'^2}{f_i^2} - \frac{f_i''}{f_i} &= h^2\lambda, \qquad i = 1, \cdots, n,
\end{aligned}
\tag{2.6}
$$

respectively.

Since $G/H \times [0,1]$ is a manifold with boundary, it is desirable to solve the Einstein and quasi-Einstein equations with various boundary conditions. We solve these equations with Dirichlet conditions. This involves finding a G-invariant Einstein metric $g = h^2 dr \otimes dr + g_r$ such that g_0 and g_1 coincide with fixed homogeneous metrics, or a G-invariant quasi-Einstein metric (g,u) such that g_0 and g_1 coincide with fixed homogeneous metrics and the function $u(r)$ coincides with fixed real numbers for $r = 0$ and $r = 1$.

Theorem 2.2.3. *Fix two homogeneous Riemannian metrics $\hat{g}_0$ and $\hat{g}_1$. Then there exists a one-parameter family of G-invariant Einstein metrics (g,λ) such that $\hat{g}_i = g_i$ for $i = 0,1$, i.e., there exists a one-parameter family of solutions to* (2.5) *subject to fixed but arbitrary Dirichlet conditions for the positive functions f_i.*

Theorem 2.2.4. *Fix two homogeneous Riemannian metrics $\hat{g}_0$ and $\hat{g}_1$, two numbers u_0 and u_1 and a constant λ. Then there exists a one-parameter family of G-invariant quasi-Einstein metrics (g,u) such that $\hat{g}_i = g_i$ and $u(i) = u_i$ for $i = 0,1$, i.e., there exists a one-parameter family of solutions to* (2.6) *subject to fixed but arbitrary Dirichlet conditions for u and the positive functions f_i.*

Remark 2.2.5. *Although we do not address this, it is natural to ask how the Einstein and quasi-Einstein metrics found in Theorems 2.2.3 and 2.2.4 behave under changes to the Dirichlet conditions. In particular, if we take a sequence of metrics on the boundary of $G/H \times [0,1]$ that become degenerate, does the corresponding sequence of Einstein or quasi-Einstein metrics converge to some other Einstein or quasi-Einstein metric? If such a limit exists, it cannot be a Riemannian metric on $G/H \times [0,1]$, but could be a metric on some other manifold, so it is conceivable that one could use this approach to produce Einstein and quasi-Einstein metrics on other manifolds.*

These results are proved with Schauder degree theory in Chapter 4.

The proof of Theorem 2.1.3 appearing in Chapter 3 originally appeared in Section 5 of the following manuscript:

1. [18] **T. Buttsworth**, A. Pulemotov, Y.A. Rubinstein, and W. Ziller, On the Ricci iteration for homogeneous metrics on spheres and projective spaces, submitted.

Timothy Buttsworth was solely responsible for the initial concept of the proof and was responsible for the majority of the theoretical derivations. The four authors were equally responsible for the writing and proof-reading.

Chapter 3

Brouwer degree theory

Degree theory provides a method of proving existence of solutions to equations via topological considerations. In this chapter, we look at Brouwer degree theory, which concerns equations on finite-dimensional spaces, and use this theory to prove Theorem 2.1.3.

3.1 The intermediate value theorem: one-dimensional Brouwer degree theory

To start our investigation into the Brouwer degree, we consider a continuous function $f : [a,b] \to \mathbb{R}$ and some $y \in \mathbb{R}$. The intermediate value theorem asserts that as long as $f(a) - y$ and $f(b) - y$ have opposite signs, the equation $f(x) = y$ must be solvable for some $x \in (a,b)$. Thus, at the very least, the information provided by the function at the boundary gives us a rough insight into the question of existence of solutions to $f(x) = y$ for $x \in (a,b)$. This motivates the following definition for functions of one variable.

Definition 3.1.1. *Choose $y \in \mathbb{R}$ and a continuous function $f : [a,b] \to \mathbb{R}$ such that $f^{-1}(y) \cap \{a,b\} = \emptyset$. The Brouwer degree of f is given by*

$$deg(f,(a,b),y) = \begin{cases} 1 & \text{if } f(a) - y < 0 < f(b) - y, \\ -1 & \text{if } f(a) - y > 0 > f(b) - y, \\ 0 & \text{otherwise}. \end{cases}$$

The following lemma describes some useful information about the Brouwer degree.

Lemma 3.1.2. *Let $f : [a,b] \to \mathbb{R}$ be continuous and choose $y \in \mathbb{R}$ so that $f(a) \neq y$ and $f(b) \neq y$. Then the Brouwer degree satisfies the following properties:*

a) If $f : [a,b] \to [a,b]$ is the identity map and $y \in (a,b)$, then $deg(f,(a,b),y) = 1$.

b) If $c \in (a,b)$ is such that $f(c) \neq y$, then

$$deg(f,(a,b),y) = deg(f,(a,c),y) + deg(f,(c,b),y).$$

c) If $H : [0,1] \times [a,b] \to \mathbb{R}$ is a continuous function such that for any $t \in [0,1]$ and $x \in \{a,b\}$ we have $H(t,x) \neq y$, then $deg(H(t,\cdot),(a,b),y)$ is independent of t.

d) If f is continuously differentiable and $f'(x) \neq 0$ for all $x \in (a,b)$ with $f(x) = y$, then $deg(f,(a,b),y) = \sum_{x \in f^{-1}(y)} sign(f'(x))$.

Proof. Properties a) and b) follow immediately from the definition of the Brouwer degree. Property c) follows from the fact that if H is continuous and $H(t,x) \neq y$ for each $(t,x) \in [0,1] \times \{a,b\}$, then the signs of $H(t,a) - y$ and $H(t,b) - y$ are independent of t.

For property d), note that if there is no $x \in (a,b)$ satisfying $f(x) = y$, then $deg(f,(a,b),y) = 0$ by the Intermediate Value Theorem. If there is precisely one $x \in (a,b)$ with $f(x) = y$, and $f'(x) > 0$, then $f(x,b)$ and the Intermediate Value Theorem implies that $f(b) - y > 0$ and $f(a) - y < 0$ so $deg(f,(a,b),y) = 1$. A similar argument shows that if there is precisely one $x \in (a,b)$ with $f(x) = y$, and $f'(x) < 0$, then $deg(f,(a,b),y) = -1$.

If there are multiple values of $x \in (a,b)$ with $f(x) = y$, note that there can only be finitely many since f is differentiable and $f'(x) \neq 0$ for each such x. Then we can break (a,b) up into finitely many intervals, each containing exactly one solution of $f(x) = y$, so we can determine the degree of f on each of these intervals as before. Property d) then follows from property b). □

3.2 Finite-dimensional Brouwer degree theory

Now we turn to the task of extending the definition of degree to higher dimensions. Much of this material appears in pages 185–225 of [2]. Since the boundary of an open set in higher dimensions is not going to be simply two points, it is not as straightforward to define the degree using only information at the boundary. On the other hand, property *d*) of Lemma 3.1.2 can be extended to higher-dimensional functions.

Definition 3.2.1. *Let $\Omega \subset \mathbb{R}^d$ be some bounded, convex and open domain, let $y \in \mathbb{R}^d$, and let $f : \bar{\Omega} \to \mathbb{R}$ be a continuously differentiable function such that $y \notin f(\partial\Omega)$. We denote with $sign(J(f))(x)$ the sign of the Jacobian determinant of f evaluated at x. If $sign(J(f))(x) \neq 0$ for all $x \in f^{-1}(y)$, then y is said to be a regular value of f. The Brouwer degree of f at a regular value y is defined to be*

$$deg(f,\Omega,y) = \sum_{x \in f^{-1}(y)} sign(J(f))(x).$$

Note that the sum $\sum_{x \in f^{-1}(y)} \text{sign}(J(f))(x)$ in Definition 3.2.1 is finite because $\text{sign}(J(f))(x) \neq 0$ for all $x \in f^{-1}(y)$. There are two main problems associated with this definition of degree. One is that we wish the degree to be *topological*, so concepts like differentiability and regular values should not appear. The other is that Definition 3.2.1 does not allow for easy computation of the degree.

The problem of defining the degree for functions with weaker differentiability properties is overcome in two steps. The first of these is to define the degree for functions that are merely assumed to be continuously differentiable, without the requirement that y is a regular value of f. This is done using an integral form of degree.

Theorem 3.2.2. *Let f and Ω be as in Definition 3.2.1. Choose a d-cube C such that $C \cap f(\partial\Omega) = \emptyset$, and a function $\phi \in C_c^\infty(C:\mathbb{R})$ satisfying $\int_C \phi = 1$. Then for every regular $y \in C$,*

$$deg(f,\Omega,y) = \int_\Omega \phi(f(x))J(f)(x)dx.$$

To show that the degree $deg(f,\Omega,y)$ can be defined for a non-regular value $\tilde{y}$, choose a cube C such that $\tilde{y} \in C$ and $C \cap f(\Omega) = \emptyset$. By Theorem 3.2.2, the degree $deg(f,\Omega,y)$ is independent of the regular value $y \in C$. By Sard's Theorem, the regular values y are dense in C, so the degree can be extended to all values in C.

We are almost in a position to define the degree for all continuous functions. Doing so will require the following theorem which can be proved with the help of Theorem 3.2.2.

Theorem 3.2.3. *The Brouwer degree satisfies the following properties:*

(i) If $y \in \Omega$ and I is the identity map, then $deg(I,\Omega,y) = 1$.

(ii) If $\Omega_1 \subset \Omega$ and $\Omega_2 \subset \Omega$ are open and convex sets such that $\Omega_1 \cap \Omega_2 = \emptyset$, $\bar{\Omega}_1 \cup \bar{\Omega}_2 = \bar{\Omega}$ and $y \notin f(\partial\Omega_1) \cup f(\partial\Omega_2)$, then $deg(f,\Omega,y) = deg(f,\Omega_1,y) + deg(f,\Omega_2,y)$.

(iii) If $H : [0,1] \times \bar{\Omega} \to \mathbb{R}^d$ is differentiable and $y \notin H(t,\partial\Omega)$, then $deg(H(t,\cdot),\Omega,y)$ is independent of t.

(iv) If $deg(f,\Omega,y) \neq 0$, then there exists $x \in \Omega$ such that $f(x) = y$.

With this theorem, we define the Brouwer degree $deg(f,\Omega,y)$ for a continuous function $f : \bar{\Omega} \to \mathbb{R}^d$ with $y \notin f(\partial\Omega)$ to be the same as $deg(\tilde{f},\Omega,y)$, where $\tilde{f} : \Omega \to \mathbb{R}^d$ is any continuously differentiable function such that $||\tilde{f} - f||_{C^0(\bar{\Omega})} \leq \frac{\inf_{x\in\partial\Omega}|f(x)-y|}{2}$. The Stone-Weierstrass theorem implies that such a function exists because $\inf_{x\in\partial\Omega}|f(x) - y| > 0$. Furthermore, the definition is independent of the choice of $\tilde{f}$ because of condition (iii) of Theorem 3.2.3. Indeed, if f^* is any other suitable differentiable function, then the function $H : [0,1] \times \bar{\Omega} \to \mathbb{R}^d$ with $H(t,x) = t\tilde{f}(x) + (1-t)f^*(x)$ does not achieve a value of y for any $(t,x) \in [0,1] \times \partial\Omega$, so we conclude that $deg(\tilde{f},\Omega,y) = deg(f^*,\Omega,y)$. If we extend the Brouwer degree to continuous functions in this way, the degree still satisfies the four properties listed in Theorem 3.2.3.

3.3 Proof of Theorem 2.1.3

Recall that to prove Theorem 2.1.3, we need to fix positive constants $\{T_i\}_{i=1}^3$ satisfying $\frac{1}{T_i} < 2n+4$, and find positive numbers x_1, x_2, x_3 and c satisfying equation (2.3), which is equivalent to

$$\begin{aligned} c &= (4n+8) - 2(x_1+x_2+x_3), \\ x_2x_3cT_1 &= \lambda x_1^2x_2x_3 + 2(x_1^2 - (x_2-x_3)^2), \\ x_1x_3cT_2 &= \lambda x_2^2x_1x_3 + 2(x_2^2 - (x_1-x_3)^2), \\ x_1x_2cT_3 &= \lambda x_3^2x_2x_1 + 2(x_3^2 - (x_1-x_2)^2), \end{aligned} \tag{3.1}$$

with $\lambda = 4n$. We solve this equation for $\lambda = 4n$ by using Brouwer degree theory. This involves treating λ as a parameter that we allow to range from 0 to $4n$.

To start, we note that when $\lambda = 0$, the last three equations of (3.1) are the same as those of the problem of prescribed Ricci curvature problem for left-invariant Riemannian metrics on $SO(3)$. This problem has been treated by Hamilton in [34]. In particular, the following result is a restatement of Theorem 6.1 in [34].

Lemma 3.3.1. *Set $\lambda = 0$. For any positive triple (T_1, T_2, T_3), there exists a unique c_0 such that when $c = c_0$, the last three equations of (3.1) are solvable for $x_1, x_2, x_3 > 0$. The solution (x_1, x_2, x_3) is unique up to scaling.*

Lemma 3.3.2. *Assume the c_0 referred to in Lemma 3.3.1 is not $4n+8$. Then there exists a bounded convex open subset Ω of $(0,\infty)^4$ such that for $\lambda \in [0,4n]$, any solution (x_1,x_2,x_3,c) of (3.1) lies in Ω.*

Proof. Assume to the contrary that no such set exists. Then there is a sequence $\lambda^{(i)} \in [0,4n]$ depending monotonically on i with a corresponding sequence $\left(x_1^{(i)}, x_2^{(i)}, x_3^{(i)}, c^{(i)}\right) \in (0,\infty)^4$ of solutions to (3.1) such that all of the variables are monotone, and at least one of the variables goes to 0 or ∞. For the remainder of the proof, we suppress reference to i to simplify notation. The first equation in (3.1) shows that no variable can go to ∞. We will consider two cases, first that $c \to 0$, and second, that at least one of x_1, x_2 or x_3 goes to 0 and c does not converge to 0. We show that we get a contradiction in both cases.

First Case. If $c \to 0$, then passing to the limits of (3.1), we find that $x_1 \to y_1$, $x_2 \to y_2$, $x_3 \to y_3$ and $\lambda \to \mu$, where y_i and μ are non-negative numbers solving

$$\begin{aligned}
0 &= (4n+8) - 2(y_1+y_2+y_3),\\
0 &= \mu y_1^2 y_2 y_3 + 2(y_1^2 - (y_2-y_3)^2),\\
0 &= \mu y_2^2 y_1 y_3 + 2(y_2^2 - (y_1-y_3)^2),\\
0 &= \mu y_3^2 y_1 y_2 + 2(y_3^2 - (y_1-y_2)^2).
\end{aligned} \tag{3.2}$$

First, we claim that at least two of y_1, y_2, y_3 are identical. To see this, note that by taking differences of the last three equations of (3.2), we find

$$\begin{aligned}
0 &= (y_1 - y_2)\left(\mu y_1 y_2 y_3 + 4(y_1 + y_2 - y_3)\right),\\
0 &= (y_2 - y_3)\left(\mu y_1 y_2 y_3 + 4(y_2 + y_3 - y_1)\right),\\
0 &= (y_3 - y_1)\left(\mu y_1 y_2 y_3 + 4(y_1 + y_3 - y_2)\right).
\end{aligned} \tag{3.3}$$

If all of y_1, y_2, y_3 are distinct, then at least one of y_1, y_2, y_3 is positive, and (3.3) implies that

$$\begin{aligned}
0 &= \left(\mu y_1 y_2 y_3 + 4(y_1 + y_2 - y_3)\right),\\
0 &= \left(\mu y_1 y_2 y_3 + 4(y_2 + y_3 - y_1)\right),\\
0 &= \left(\mu y_1 y_2 y_3 + 4(y_1 + y_3 - y_2)\right).
\end{aligned} \tag{3.4}$$

By adding these three equations up, we find that the non-negative numbers y_1, y_2, y_3 and μ satisfy $3\mu y_1 y_2 y_3 + 4(y_1 + y_2 + y_3) = 0$, which is a contradiction since one of y_1, y_2, y_3 is positive.

We now know that at least two of y_1, y_2, y_3 are identical, so we assume without loss of generality that $y_2 = y_3$. Then since μ, y_1, y_2 and y_3 are all non-negative, the second equation of (3.2) implies that $y_1 = 0$, and the first equation implies that $y_2 = y_3 > 0$. Now by dividing the third and fourth equations of (3.1) by x_1x_3 and x_1x_2 respectively, we see that

$$cT_2 = \lambda x_2^2 + 2\Big(\frac{x_2^2}{x_1x_3} - \frac{x_1}{x_3} - \frac{x_3}{x_1} + 2\Big),$$
$$cT_3 = \lambda x_3^2 + 2\Big(\frac{x_3^2}{x_1x_2} - \frac{x_1}{x_2} - \frac{x_2}{x_1} + 2\Big).$$

We rearrange to find

$$cT_2 - \lambda x_2^2 - 4 + 2\frac{x_1}{x_3} = 2\Big(\frac{x_2^2}{x_1x_3} - \frac{x_3}{x_1}\Big),$$
$$cT_3 - \lambda x_3^2 - 4 + 2\frac{x_1}{x_2} = 2\Big(\frac{x_3^2}{x_1x_2} - \frac{x_2}{x_1}\Big).$$

Since $x_1 \to y_1 = 0$, $c \to 0$, $\lambda \geq 0$ and $x_2, x_3 \to y_2 = y_3 > 0$, we can take limits to find that

$$2\lim\Big(\frac{x_2^2}{x_1x_3} - \frac{x_3}{x_1}\Big) = -\mu y_2^2 - 4 < 0,$$
$$2\lim\Big(\frac{x_3^2}{x_1x_2} - \frac{x_2}{x_1}\Big) = -\mu y_3^2 - 4 < 0.$$

We deduce that eventually, both $\frac{x_2^2}{x_1x_3} - \frac{x_3}{x_1}$ and $\frac{x_3^2}{x_1x_2} - \frac{x_2}{x_1}$ must be negative. Thus $x_2^2 - x_3^2 < 0$ and $x_3^2 - x_2^2 < 0$, which is a contradiction.

Second Case. Now c converges to some positive number, but at least one of x_1, x_2, x_3 is converging to 0. To start, assume that $x_1 \to 0$. The third equation of (3.1) implies that $x_2 - x_3 \to 0$. The second equation of (3.1) then implies that $x_2x_3c \to 0$. Since c does not converge to 0, we must have both x_2 and x_3 converging to 0 as well as x_1. If instead of assuming $x_1 \to 0$ we assume that $x_2 \to 0$ or $x_3 \to 0$, we would again conclude that all three of x_1, x_2, x_3 are converging to 0.

Since $x_1, x_2, x_3 \to 0$, the first equation of (3.1) implies that $c \to 4n+8$. Rewrite the second, third and fourth equations as

$$cT_1 = \lambda x_1^2 + 2z_2z_3, \qquad cT_2 = \lambda x_2^2 + 2z_1z_3, \qquad cT_3 = \lambda x_3^2 + 2z_1z_2, \tag{3.5}$$

where $z_1 = \frac{x_2+x_3-x_1}{x_1}$, $z_2 = \frac{x_1+x_3-x_2}{x_2}$, $z_3 = \frac{x_1+x_2-x_3}{x_3}$, and we can assume that these numbers change monotonically as well. For each $i = 1, 2, 3$, $\lambda x_i^2 \to 0$ and $cT_i \to (4n+8)T_i > 0$, so (3.5) implies that all three of z_1, z_2, z_3 are bounded. Indeed, suppose z_i is unbounded for some i and choose j and k so that i, j and k are pairwise distinct. Then the equations for T_j and T_k imply that both z_j and z_k are converging to 0, which contradicts the equation for T_i.

Since we now know that z_1, z_2 and z_3 are bounded, we deduce that $\frac{x_i}{x_j}$ is bounded for each i and j.

Now write (3.5) as

$$\begin{aligned} cT_1 &= \lambda x_1^2 + \frac{2((dx_1)^2 - (dx_2 - dx_3)^2)}{dx_2 dx_3}, \\ cT_2 &= \lambda x_2^2 + \frac{2((dx_2)^2 - (dx_1 - dx_3)^2)}{dx_1 dx_3}, \\ cT_3 &= \lambda x_3^2 + \frac{2((dx_3)^2 - (dx_1 - dx_2)^2)}{dx_2 dx_1}, \end{aligned} \tag{3.6}$$

where $d = 1/x_1$ (again dropping reference to the superscript). By taking a subsequence, we can assume that dx_2 and dx_3 are monotone. Since $\frac{x_i}{x_j}$ is a bounded sequence for each i and j, we know that dx_2 and dx_3 converge to some positive numbers. Taking limits and using the fact that $c \to 4n+8$, we see that the numbers dx_i converge to a solution of the first three equations of (3.1) with $c_0 = 4n+8$, which contradicts our assumption on c_0. □

By moving everything to the right-hand side, we can write (3.1) as $f(\lambda, x_1, x_2, x_3, c) = 0$, where $f : [0,4n] \times (0,\infty)^4 \to \mathbb{R}^4$ is continuous. An immediate consequence of Lemma 3.3.2 is the following.

Corollary 3.3.3. *If $c_0 \neq 4n+8$, then for any Ω satisfying the conclusion of Lemma 3.3.2, we have*

$$deg(f(4n,\cdot),\Omega,0) = deg(f(0,\cdot),\Omega,0).$$

Proof. The result follows from the homotopy invariance of the Brouwer degree (condition (*iii*) of Theorem 3.2.3). □

From now on, we assume that $c_0 \neq 4n+8$, and we fix an Ω satisfying the conclusion of Lemma 3.3.2. We will evaluate $deg(f(0,\cdot),\Omega,0)$.

Lemma 3.3.4. *If $c_0 > 4n+8$, then $deg(f(0,\cdot),\Omega,0) = 0$.*

Proof. The equation $f(0,\cdot) = 0$ consists of the equation discussed in Lemma 3.3.1, alongside $c = 4n+8-2(x_1+x_2+x_3)$. By Lemma 3.3.1, c coincides with c_0. However, then any solution satisfies $2(x_1+x_2+x_3) = 4n+8-c_0 < 0$, a contradiction. Therefore, in Ω, no solution of $f(0,x_1,x_2,x_3,c) = 0$ exists, and we conclude that $deg(f(0,\cdot),\Omega,0) = 0$. □

Similarly, we can prove the following.

Lemma 3.3.5. *If $c_0 < 4n+8$, $deg(f(0,\cdot),\Omega,0) \neq 0$.*

Proof. By Lemma 3.3.1, if $\lambda = 0$, we have a unique solution of the last three equations of (3.1) up to scaling of x_i. Since $c_0 < 4n+8$, x_i can be scaled uniquely so that the first equation of (3.1) is satisfied. This tells us that a solution of $f(0,\cdot) = 0$ exists and is unique in $(\mathbb{R}^+)^4$. By looking at the proof of Lemma 3.3.1, it is straightforward to demonstrate that the derivative of $f(0,\cdot)$ is non-degenerate at this solution, so $deg(f(0,\cdot),\Omega,0) \neq 0$. □

Corollary 3.3.3 together with Lemma 3.3.5 demonstrates that $deg(f(4n,\cdot),\Omega,0) \neq 0$, so property (iv) of Theorem 3.2.3 implies that there is a solution of $f(4n,x_1,x_2,x_3,c)=0$ in Ω provided $c_0 < 4n+8$. This corresponds to a solution of (3.1) (and also (2.3)). The proof of Theorem 2.1.3 is then concluded with the following lemma.

Lemma 3.3.6. *If* (x_1,x_2,x_3,c_0) *solve the last three equations of system* (3.1) *with* $\lambda = 0$*, and* $\frac{1}{T_i} < 2n+4$ *for all i, then* $c_0 < 4n+8$.

Proof. By dividing each of the last three equations in (3.1) by $x_1x_2x_3$ and adding, we obtain:

$$\begin{aligned} c_0\sum_{i=1}^{3}\frac{T_i}{x_i} &= 4\sum_{i=1}^{3}\frac{1}{x_i} - 2\frac{x_1}{x_2x_3} - 2\frac{x_2}{x_1x_3} - 2\frac{x_3}{x_1x_2} \\ &= 4\sum_{i=1}^{3}\frac{1}{x_i} - \frac{1}{x_1}\Big(\frac{x_2}{x_3}+\frac{x_3}{x_2}\Big) - \frac{1}{x_2}\Big(\frac{x_1}{x_3}+\frac{x_3}{x_1}\Big) - \frac{1}{x_3}\Big(\frac{x_1}{x_2}+\frac{x_2}{x_1}\Big) \\ &\leq 4\sum_{i=1}^{3}\frac{1}{x_i} - 2\sum_{i=1}^{3}\frac{1}{x_i} = 2\sum_{i=1}^{3}\frac{1}{x_i} \end{aligned}$$

since $a+1/a \geq 2$ for $a>0$. This implies the claim. □

The condition in Theorem 2.1.3 is not necessary. In fact, its proof shows that one has the following stronger statement:

Theorem 3.3.7. *There exists an* $Sp(n+1)$*-invariant metric g such that* $Ricg = cT$ *for some* $c>0$ *if one has*

$$c_0 = c_0(T_1,T_2,T_3) < 4n+8,$$

where c_0 *is the number depending on* T_1,T_2,T_3 *described in Lemma 3.3.1.*

It is conceivable that $c_0 < 4n+8$ is necessary and sufficient for the solvability of $Ric(g) = cT$.

Parts of the following sole-authored publications appear in Chapter 4:

1. [16] **T. Buttsworth**, The Dirichlet problem for Einstein metrics on cohomogeneity one manifolds, *Ann. Glob. Anal. Geom.*, 54, 1, 2018.

2. [17] **T. Buttsworth**, Cohomogeneity-one quasi-Einstein metrics, *J. Math. Anal. Appl.*, 470, 1, 2019.

Chapter 4

Schauder degree theory

In the previous chapter we examined the construction of the Brouwer degree and found it to be useful when solving equations in $\mathbb{R}^d$. However, the study of differential equations usually occurs on infinite-dimensional Banach spaces, so the theory needs to be extended before it can be applicable to the proofs of Theorems 2.2.3 and 2.2.4. In this chapter, we recall some useful properties of the Schauder degree and use these properties to prove Theorems 2.2.3 and 2.2.4. Our use of the Schauder degree is only perturbative, so in this chapter, we also provide examples to demonstrate that we cannot expect to use the Schauder degree in the non-perturbative setting.

4.1 Properties of the Schauder degree

The information in this section is standard and can be found in, for example, Section 2 of [38]. Let X be a Banach space, $U \subset X$ a bounded, open and convex subset, $I : X \to X$ the identity mapping and $f : \bar{U} \to X$ compact. Also choose $y \in X \setminus (I - f)(\partial U)$. The Schauder degree is calculated by approximating the compact function f by a sequence of maps f_n whose images lie inside finite dimensional subspaces. Indeed, since f is compact, there exists an $\varepsilon > 0$ so that $B_\varepsilon(y) \cap f(\partial U) = \emptyset$ and we can choose a continuous map $R : \bar{U} \to X_0 \subset X$ so that X_0 is finite-dimensional and $|f(x) - R(x)| < \frac{\varepsilon}{2}$ for each $x \in \bar{U}$. The Schauder degree $deg(I - f, U, y)$ can then be defined to be equal to the Brouwer degree $deg((I + R)|_{U \cap X_0}, U \cap X_0, y)$. The properties of the Brouwer degree imply that this definition is independent of the choice of R.

The following theorem tells us the key properties of the Schauder degree that will be useful in our work on solving geometric equations.

Theorem 4.1.1. *The Schauder degree has the following properties:*

(i) Suppose that the Frechet derivative $f'(x)$ exists and $I - f'(x)$ is invertible for each $x \in (I - f)^{-1}(y)$. Then the Schauder degree of f satisfies

$$deg(I - f, U, y) = \sum_{x \in (I-f)^{-1}(y)} (-1)^{\sigma(x)},$$

where $\sigma(x)$ is the sum of the algebraic multiplicities of the eigenvalues of $f'(x)$ contained in $(1,\infty)$.

(ii) If $H:[0,1]\times\bar{U}\to X$ is compact and $y\notin(I-H(t,\cdot))(\partial U)$ for each $t\in[0,1]$, then $deg(I-H(t,\cdot),U,y)$ is independent of t.

(iii) If $deg(I-f,U,y)\neq 0$, then there exists $x\in U$ such that $x-f(x)=y$.

Theorem 4.1.1 provides us with a framework for proving existence of solutions to certain differential equations. Indeed, if we can write our equation as $(I-f)(x)=y$, then the existence of a solution follows from constructing a U such that $deg(I-f,U,y)\neq 0$. It is generally hard to do this just using property (i) of Theorem 4.1.1, so instead, we can attempt to find a function H satisfying property (ii) such that $H(1,\cdot)=f$, and such that $H(0,\cdot)$ is simpler. If $H(0,\cdot)$ is simple enough, we can compute $deg(I-H(0,\cdot),U,y)$, which if non-zero, will imply that $(I-f)(x)=y$ has a solution.

4.2 Proof of Theorem 2.2.3

To prove Theorem 2.2.3, we need to fix two arrays of positive numbers $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$, and find functions f_i, as well as constants λ and $h>0$ satisfying

$$\begin{aligned} -\sum_{k=1}^n d_k\frac{f_k''}{f_k} &=\lambda, \\ \frac{h^2\beta_i}{2f_i^2}+h^2\sum_{k,l=1}^n \gamma_{ik}^l\frac{f_i^4-2f_k^4}{4f_i^2f_k^2f_l^2}-\frac{f_i'}{f_i}\sum_{k=1}^n d_k\frac{f_k'}{f_k}+\frac{f_i'^2}{f_i^2}-\frac{f_i''}{f_i} &=\lambda, \\ f_i(0)=a_i,\qquad f_i(1)=b_i,\qquad & i=1,\cdots,n. \end{aligned} \tag{4.1}$$

Note that the first two equations of (4.1) are the same as (2.5), except we have absorbed the constant h^2 into the constant λ, and so now the right-hand sides of the first two equations are λ instead of $h^2\lambda$. Before solving this problem, we show with the following lemma that we can replace (4.1) with an equation of a form allowing easier use of Schauder degree theory. The key part of the proof of the following lemma is a computation in (4.2), which is essentially the second contracted Bianchi identity.

Lemma 4.2.1. *There exists a compact function*

$$H:[0,1]\times\mathbb{R}\times C^1([0,1]:(\mathbb{R}^n)^+)\to\mathbb{R}\times C^1([0,1]:\mathbb{R}^n)$$

such that $(\lambda,f)=H(h,\lambda,f)$ if and only if (h,λ,f) solves (4.1).

Proof. Let

$$L=diag\Big(\underbrace{\frac{f_1'}{f_1},\cdots,\frac{f_1'}{f_1}}_{d_1\text{ times}},\cdots,\underbrace{\frac{f_n'}{f_n},\cdots,\frac{f_n'}{f_n}}_{d_n\text{ times}}\Big)$$

and similarly, let R be the diagonal matrix with entries $\frac{h^2\beta_i}{2f_i^2}+h^2\sum_{k,l=1}^n\gamma_{ik}^l\frac{f_i^4-2f_k^4}{4f_i^2f_k^2f_l^2}$, each occurring d_i times. Let $y(r)=tr(L')+tr(L^2)+\lambda$ and assume that the second equation of (4.1) holds for each i.

Then $y(r) = tr(L^2) - tr(L)^2 + (1-d)\lambda + tr(R)$, where $d = \sum_{i=1}^n d_i$, and

$$\begin{aligned} \frac{1}{2}y'(r) &= tr(LL') - tr(L)tr(L') + \frac{1}{2}tr(R)' \\ &= tr(L(-tr(L)L + R - \lambda I)) + tr(L)^3 - tr(L)tr(R) + dtr(L)\lambda + \frac{1}{2}tr(R)' \\ &= tr(L)\left((d-1)\lambda - tr(L^2) + tr(L)^2 - tr(R)\right) + tr(LR) + \frac{1}{2}tr(R)' \\ &= -tr(L)y(r). \end{aligned} \tag{4.2}$$

To see that $tr(LR) + \frac{1}{2}tr(R)' = 0$, note that

$$tr(R) = \frac{h^2}{2}\sum_{i=1}^n \frac{d_i\beta_i}{f_i^2} - \frac{h^2}{4}\sum_{i,k,l=1}^n d_i\gamma_{ik}^l \frac{f_l^2}{f_i^2 f_k^2}$$

since $d_i\gamma_{ik}^l$ is symmetric in i,k,l. We then see that

$$\frac{\partial tr(R)}{\partial f_i} = -\frac{2d_i}{f_i}\left(\frac{h^2\beta_i}{2f_i^2} + h^2\sum_{k,l=1}^n \gamma_{ik}^l \frac{f_i^4 - 2f_k^4}{4f_i^2 f_k^2 f_l^2}\right),$$

so

$$\begin{aligned} tr(LR) + \frac{1}{2}tr(R)' &= tr(LR) - \sum_{i=1}^n \frac{d_i f_i'}{f_i}\left(\frac{h^2\beta_i}{2f_i^2} + h^2\sum_{k,l=1}^n \gamma_{ik}^l \frac{f_i^4 - 2f_k^4}{4f_i^2 f_k^2 f_l^2}\right) \\ &= 0 \end{aligned}$$

as required. This computation implies that if the second equation of (4.1) holds for each i and the first equation holds at some point, say, $r = 0$, then $y(0) = 0$, and so $y(r) = 0$ for all $r \in [0,1]$. Therefore, solving (4.1) is equivalent to solving

$$\begin{aligned} -\left(\sum_{k=1}^n d_k \frac{f_k''}{f_k}\right)\Bigg|_{r=0} &= \lambda, \\ \frac{h^2\beta_i}{2f_i^2} + h^2\sum_{k,l=1}^n \gamma_{ik}^l \frac{f_i^4 - 2f_k^4}{4f_i^2 f_k^2 f_l^2} - \frac{f_i'}{f_i}\sum_{k=1}^n d_k\frac{f_k'}{f_k} + \frac{f_i'^2}{f_i^2} - \frac{f_i''}{f_i} &= \lambda, \\ f_i(0) = a_i, \qquad f_i(1) = b_i, \qquad i &= 1, \cdots, n. \end{aligned} \tag{4.3}$$

By combining the equations of (4.3), it is clear that solving (4.3) is also equivalent to solving

$$\begin{aligned} \sum_{i=1}^n d_i\left(\frac{h^2\beta_i}{2f_i^2} + \sum_{k,l=1}^n h^2\gamma_{ik}^l \frac{f_i^4 - 2f_k^4}{4f_i^2 f_k^2 f_l^2} - \frac{f_i'}{f_i}\sum_{k=1}^n d_k\frac{f_k'}{f_k} + \frac{f_i'^2}{f_i^2}\right)\Bigg|_{r=0} &= (d-1)\lambda, \\ \frac{h^2\beta_i}{2f_i^2} + h^2\sum_{k,l=1}^n \gamma_{ik}^l \frac{f_i^4 - 2f_k^4}{4f_i^2 f_k^2 f_l^2} - \frac{f_i'}{f_i}\sum_{k=1}^n d_k\frac{f_k'}{f_k} + \frac{f_i'^2}{f_i^2} - \frac{f_i''}{f_i} &= \lambda, \\ f_i(0) = a_i, \qquad f_i(1) = b_i, \qquad i &= 1, \cdots, n. \end{aligned} \tag{4.4}$$

We rearrange to find

$$\begin{aligned} \frac{1}{d-1}\sum_{i=1}^n d_i\left(\frac{h^2\beta_i}{2f_i^2} + \sum_{k,l=1}^n h^2\gamma_{ik}^l \frac{f_i^4 - 2f_k^4}{4f_i^2 f_k^2 f_l^2} - \frac{f_i'}{f_i}\sum_{k=1}^n d_k\frac{f_k'}{f_k} + \frac{f_i'^2}{f_i^2}\right)\Bigg|_{r=0} &= \lambda, \\ f_i\left(\frac{h^2\beta_i}{2f_i^2} + h^2\sum_{k,l=1}^n \gamma_{ik}^l \frac{f_i^4 - 2f_k^4}{4f_i^2 f_k^2 f_l^2} - \frac{f_i'}{f_i}\sum_{k=1}^n d_k\frac{f_k'}{f_k} + \frac{f_i'^2}{f_i^2} - \lambda\right) &= f_i'', \\ f_i(0) = a_i, \qquad f_i(1) = b_i, \qquad i &= 1, \cdots, n. \end{aligned} \tag{4.5}$$

If we multiply both sides of the second equation of (4.6) by the Green's function

$$G(x,r) = \begin{cases} x(r-1) & \text{if } 0 \le x \le r \le 1, \\ (x-1)r & \text{if } 0 \le r < x \le 1 \end{cases}$$

and integrate with respect to r, we find that

$$\lambda = \frac{1}{d-1}\sum_{i=1}^{n} d_i \left(\frac{h^2\beta_i}{2f_i^2} + \sum_{k,l=1}^{n} h^2 \gamma_{ik}^l \frac{f_i^4 - 2f_k^4}{4f_i^2 f_k^2 f_l^2} - \frac{f_i'}{f_i}\sum_{k=1}^{n} d_k \frac{f_k'}{f_k} + \frac{f_i'^2}{f_i^2} \right)\Bigg|_{r=0},$$
$$f_i(x) = a_i(1-x) + b_i x + \int_0^1 G(x,r) f_i \left(\frac{h^2\beta_i}{2f_i^2} + h^2 \sum_{k,l=1}^{n} \gamma_{ik}^l \frac{f_i^4 - 2f_k^4}{4f_i^2 f_k^2 f_l^2} - \frac{f_i'}{f_i}\sum_{k=1}^{n} d_k \frac{f_k'}{f_k} + \frac{f_i'^2}{f_i^2} - \lambda \right) dr. \tag{4.6}$$

We then define $H(h,\lambda,f)$ to be the element of $\mathbb{R} \times C^1([0,1]:\mathbb{R}^n)$ with $\mathbb{R}$ component given by

$$\frac{1}{d-1}\sum_{i=1}^{n} d_i \left(\frac{h^2\beta_i}{2f_i^2} + \sum_{k,l=1}^{n} h^2 \gamma_{ik}^l \frac{f_i^4 - 2f_k^4}{4f_i^2 f_k^2 f_l^2} - \frac{f_i'}{f_i}\sum_{k=1}^{n} d_k \frac{f_k'}{f_k} + \frac{f_i'^2}{f_i^2} \right)\Bigg|_{r=0},$$

and the ith $C^1([0,1]:\mathbb{R})$ component given by

$$a_i(1-x) + b_i x + \int_0^1 G(x,r) f_i \left(\frac{h^2\beta_i}{2f_i^2} + h^2 \sum_{k,l=1}^{n} \gamma_{ik}^l \frac{f_i^4 - 2f_k^4}{4f_i^2 f_k^2 f_l^2} - \frac{f_i'}{f_i}\sum_{k=1}^{n} d_k \frac{f_k'}{f_k} + \frac{f_i'^2}{f_i^2} - \lambda \right) dr.$$

If $(h^{(k)},\lambda^{(k)},f^{(k)})$ is bounded in $[0,1]\times\mathbb{R}\times C^1([0,1]:(\mathbb{R}^n)^+)$, then $H((h^{(k)},\lambda^{(k)},f^{(k)}))$ is bounded in $\mathbb{R}\times C^2([0,1]:\mathbb{R}^n)$, so the Arzela-Ascoli Theorem implies that $H((h^{(k)},\lambda^{(k)},f^{(k)}))$ has a convergent subsequence in $\mathbb{R}\times C^1([0,1]:\mathbb{R}^n)$, so H is compact. □

If $h=0$, then equation (4.4) simplifies substantially, and by analysing these equations (which we do in Chapter 5), we find the following:

Lemma 4.2.2. *There exists a bounded, open and convex* $\Omega \subset \mathbb{R}\times C^1([0,1]:(\mathbb{R}^n)^+)$ *such that* $0 \notin (I - H(0,\cdot))(\partial\Omega)$ *and* $deg(I-H(0,\cdot),\Omega,0) \neq 0$.

Proof. We present the proof of this result in Section 5.1. □

Since H is compact, there exists a $K>0$ such that $0 \notin (I-H(h,\cdot))(\partial\Omega)$ for each $h<K$, so condition (ii) of Theorem 4.1.1 then implies that $deg(I-H(h,\cdot)\Omega,0) \neq 0$ for all $h<K$. The proof of Theorem 2.2.3 then follows from condition (iii) of Theorem 4.1.1.

4.3 Proof of Theorem 2.2.4

To prove Theorem 2.2.4, we need to fix two arrays of positive numbers $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$, three numbers c_0, c_1 and λ, and find functions f_i and u and a constant $h > 0$ satisfying

$$\begin{aligned} u'' - m(u')^2 - \sum_{k=1}^n d_k \frac{f_k''}{f_k} &= h^2\lambda, \\ u'\frac{f_i'}{f_i} + \frac{h^2\beta_i}{2f_i^2} + h^2 \sum_{k,l=1}^n \gamma_{ik}^l \frac{f_i^4 - 2f_k^4}{4f_i^2 f_k^2 f_l^2} - \frac{f_i'}{f_i}\sum_{k=1}^n d_k \frac{f_k'}{f_k} + \frac{f_i'^2}{f_i^2} - \frac{f_i''}{f_i} &= h^2\lambda, \\ f_i(0) = a_i, \qquad f_i(1) = b_i, \qquad i = 1, \cdots, n&, \\ u(0) = c_0, \qquad u(1) = c_1&. \end{aligned} \tag{4.7}$$

By writing $\xi = \sum_{k=1}^n d_k \ln(f_k) - u$, we arrive at

$$\begin{aligned} \xi'' + \sum_{k=1}^n d_k \frac{f_k'^2}{f_k^2} + m(\sum_{k=1}^n d_k \frac{f_k'}{f_k} - \xi')^2 &= -h^2\lambda, \\ \frac{h^2\beta_i}{2f_i^2} + h^2 \sum_{k,l=1}^n \gamma_{ik}^l \frac{f_i^4 - 2f_k^4}{4f_i^2 f_k^2 f_l^2} - \xi' \sum_{k=1}^n d_k \frac{f_k'}{f_k} + \frac{f_i'^2}{f_i^2} - \frac{f_i''}{f_i} &= h^2\lambda, \\ f_i(0) = a_i, \qquad f_i(1) = b_i, \qquad i = 1, \cdots, n&, \\ \xi(0) = \sum_{i=1}^n d_i \ln(a_i) - c_0, \qquad \xi(1) = \sum_{i=1}^n d_i \ln(b_i) - c_1&. \end{aligned} \tag{4.8}$$

Lemma 4.3.1. *There exists a compact function* $H : [0,1] \times C^1([0,1] : \mathbb{R}) \times C^1([0,1] : (\mathbb{R}^n)^+) \to C^1([0,1] : \mathbb{R}) \times C^1([0,1] : \mathbb{R}^n)$ *such that* $(\xi, f) = H(h, \xi, f)$ *if and only if* (h, ξ, f) *solves* (4.8).

Proof. Similarly to the proof of 4.2.1, the result follows by multiplying the differential equations of (4.8) by the Green's function $G(x,r)$, integrating and using the Dirichlet conditions. □

The following lemma discusses the case that $h = 0$.

Lemma 4.3.2. *There exists a bounded, open and convex* $\Omega \subset C^1([0,1] : \mathbb{R}) \times C^1([0,1] : (\mathbb{R}^n)^+)$ *such that* $deg(I - H(0,\cdot), \Omega, 0) \neq 0$.

Proof. We leave the proof of this result until Section 5.2. □

Once again, the fact that H is compact implies that there exists a $K > 0$ such that $0 \notin (I - H(h,\cdot))(\partial\Omega)$ for each $h < K$, so $deg(I - H(h,\cdot)\Omega, 0) \neq 0$ for all $h < K$ and the proof of Theorem 2.2.4 follows from condition (iii) of Theorem 4.1.1.

4.4 Non-existence for large h

We have found solutions of (4.1) and (4.7) using Schauder degree theory as a perturbation tool; we have only found solutions for small values of h. The purpose of this section is to demonstrate that we cannot expect existence results for large h. In a sense, this will demonstrate that our use of Schauder degree theory has reached its potential.

Our example of non-existence for (4.1) comes by choosing G/H to be the compact homogeneous space $\mathbb{S}^1 \times \mathbb{S}^2$ acted on transitively by $G = SO(2) \times SO(3)$. After choosing a reference $Ad(G)$-invariant metric Q on the Lie algebra of $SO(2) \times SO(3)$ we see that $n = 2$, $d_1 = 1$, $d_2 = 2$, and after computing the constants γ_{ij}^k and β_i, we see that (4.1) becomes

$$\begin{aligned} -\frac{f_1''}{f_1} - 2\frac{f_2''}{f_2} &= \lambda, \\ -\frac{f_1''}{f_1} - 2\frac{f_1' f_2'}{f_1 f_2} &= \lambda, \\ -\frac{f_2''}{f_2} - \frac{(f_2')^2}{f_2^2} - \frac{f_1' f_2'}{f_1 f_2} + \frac{h^2 \mu}{f_2^2} &= \lambda, \end{aligned} \tag{4.9}$$

alongside Dirichlet conditions for f_1 and f_2. Here, $\mu > 0$ is some number depending on the choice of Q. We can eliminate λ from these equations, and we find that

$$\begin{aligned} \frac{f_2''}{f_2} - \frac{f_1' f_2'}{f_1 f_2} &= 0, \\ \frac{f_1''}{f_1} - \frac{f_2'^2}{f_2^2} + \frac{h^2 \mu}{f_2^2} &= 0. \end{aligned} \tag{4.10}$$

Suppose we require that $f_2(0) = f_2(1) = 1$. Then there exists some r such that $f_2'(r) = 0$, and the first equation of (4.10) implies that $f_2'(r) = 0$ for all $r \in [0,1]$. Therefore, $f_2(r) = 1$ for all $r \in [0,1]$ and the second equation of (4.10) becomes $\frac{f_1''}{f_1} + h^2\mu = 0$. It is clear that this equation is not solvable for large $h > 0$ if we require $f_1 > 0$ on $[0,1]$.

Our example of non-existence for (4.7) comes by simply taking $G = SO(2)$ and H containing only the identity, so our homogeneous space G/H is the circle $\mathbb{S}^1$. In this case, (4.7) becomes

$$\begin{aligned} u'' - m(u')^2 - \frac{f_1''}{f_1} &= h^2\lambda, \\ u'\frac{f_1'}{f_1} - \frac{f_1''}{f_1} &= h^2\lambda, \end{aligned}$$

alongside Dirichlet conditions for u and f_1. By combining the two equations, we find that $u'' = m(u')^2 + \frac{f_1'}{f_1}u'$, which is an equation satisfying the maximum principle, so if we require that $u(0)$ and $u(1)$ are identical, then u is constant. Under such a situation, we find that

$$-\frac{f_1''}{f_1} = h^2\lambda,$$

which is impossible to solve with $f_1 > 0$ on $[0,1]$ if $h^2\lambda$ is large enough.

4.5 Non-uniqueness for large h

For (4.7), we also find that we can achieve non-uniqueness of solutions to the Dirichlet problem if h^2 is large enough. For example, we can choose G/H to be the sphere $\mathbb{S}^2$, with $G = SO(3)$ acting transitively with isotropy $SO(2)$. Then after assuming $m = \lambda = 0$ and $h = 1$, we use the change of variables $y = \ln(f_1)$ and $\xi = u - 2y$ to find that the equations of (4.7) become

$$\begin{aligned} \xi'' &= -2(y')^2, \\ y'' &= \beta e^{-2y} - \xi' y', \end{aligned} \tag{4.11}$$

alongside Dirichlet conditions for y and ξ. Here, $\beta > 0$ depends on the choice of reference metric on $\mathbb{S}^2$, which we can assume is 1 without loss of generality.

Theorem 4.5.1. *For some choices of $\bar{y} \in \mathbb{R}$, there exists at least two solutions of* (4.11) *satisfying $\xi(0) = 0 = \xi(1)$ and $y(0) = \bar{y} = y(1)$.*

To prove Theorem 4.5.1, we first note that since ξ does not appear explicitly in (4.11), it suffices to prove that there are multiple solutions of

$$\begin{aligned} v' &= -2(y')^2, \\ y'' &= \beta e^{-2y} - v y', \end{aligned} \tag{4.12}$$

subject to

$$y(0) = \bar{y}, \qquad y(1) = \bar{y}, \qquad \int_0^1 v = 0, \tag{4.13}$$

for some $\bar{y} \in \mathbb{R}$. To find solutions, we will use the following two lemmas.

Lemma 4.5.2. *Suppose that we have a solution of* (4.12) *on* $[0,1]$ *such that $v(\frac{1}{2}) = y'(\frac{1}{2}) = 0$. Then $y(0) = y(1)$ and $\int_0^1 v = 0$.*

Proof. Consider the functions $\tilde{y}(t) = y(1-t)$ and $\tilde{v}(t) = -v(1-t)$. We have $\tilde{v}(\frac{1}{2}) = \tilde{y}'(\frac{1}{2}) = 0$, $\tilde{y}(\frac{1}{2}) = y(\frac{1}{2})$, and the pair $(\tilde{v}, \tilde{y})$ also satisfies (4.12). Therefore, $(\tilde{v}, \tilde{y})$ must be identical to (v, y) on $[0,1]$. In particular, $y(0) = \tilde{y}(0) = y(1)$, and $v(t) = -v(1-t)$, so $\int_0^1 v = 0$. □

Lemma 4.5.3. *There exists $k \in \mathbb{R}$ such that a solution of* (4.12) *subject to the conditions*

$$v(\tfrac{1}{2}) = 0, \qquad y'(\tfrac{1}{2}) = 0, \qquad y(\tfrac{1}{2}) = k_1, \tag{4.14}$$

exists on $[0,1]$ *whenever $k_1 > k$.*

Proof. Consider the complete metric space

$$X = \{(v,y,z) \in C^0([\tfrac{1}{2},1] : \mathbb{R}^3) : ||v||_{C^0[\frac{1}{2},1]} \le R, ||y - k_1||_{C^0[\frac{1}{2},1]} \le 1, ||z||_{C^0[\frac{1}{2},1]} \le R\}$$

for some $R > 0$. The problem of solving (4.12) subject to (4.14) can alternately be formulated as finding a fixed point of $H : C^0([\frac{1}{2},1] : \mathbb{R}^3) \to C^0([\frac{1}{2},1] : \mathbb{R}^3)$, where the first, second and third components of $H(v,y,z)$ are given by

$$\begin{aligned} &\int_{\frac{1}{2}}^t -2z(s)^2 \, ds, \\ &k_1 + \int_{\frac{1}{2}}^t z(s) \, ds, \\ &\int_{\frac{1}{2}}^t e^{-2y(s)} - v(s)z(s) \, ds, \end{aligned}$$

respectively. For large k_1 and small R, H is a contraction on X. The result then follows from the Banach Fixed Point Theorem. □

By Lemma 4.5.2, we know that to find a solution of (4.12) with $\int_0^1 v = 0$, and $y(0) = y(1) = \bar{y}$, it suffices to find a solution of (4.12) with (4.14), where k_1 is chosen so that $y(1) = \bar{y}$. The following lemma demonstrates that a choice of k_1 is not always unique, completing the proof of Theorem 4.5.1.

Lemma 4.5.4. *There exists a value of $\bar{y}$ such that there are at least two values of k_1 for which a solution of* (4.12) *and* (4.14) *satisfies $y(1) = \bar{y}$.*

Proof. Let k^* be the infimum of all values of k such that a solution of (4.12) with (4.14) exists on $[0,1]$ whenever $k_1 > k$. Such a k^* exists because of Lemma 4.5.3. Then for all $k_1 \in (k^*, \infty)$, there exists a solution of (4.12) with (4.14) on $[0,1]$. Since the value of $y(1)$ depends continuously on k_1, the proof will be complete if we can demonstrate that $y(1)$ does not depend monotonically on $k_1 \in (k^*, \infty)$. We do this by ruling out certain cases.

Case 1: $y(1)$ is decreasing as k_1 increases on (k^*, ∞). However, by taking $k_1 \to \infty$, we see that $y(1) \to \infty$ as well because $y(1) \geq k_1$, which contradicts the assertion that $y(1)$ is monotone decreasing.

Case 2: $y(1)$ is increasing as k_1 increases and $k^* = -\infty$. Now $(2y'^2 - v^2)' = 4y'(e^{-2y} - vy') + 4v(y')^2 = 4y'e^{-2y}$, so from the fact that $2y'^2 - v^2 = 0$ at $t = \frac{1}{2}$ and $y'' \geq 0$ on $[0,1]$, we conclude that

$$2y'^2(t) \geq v(t)^2 \tag{4.15}$$

for all $t \in [0,1]$. We then find that

$$v' \leq -v^2. \tag{4.16}$$

This implies that as $k_1 \to -\infty$, $v(\frac{7}{8})$ remains bounded from below, otherwise (4.16) implies that v blows up before t gets to 1. Now using the fact that $y'' \geq 0$ on $[0,1]$, we see that

$$\begin{aligned} v(\tfrac{7}{8}) &= -\int_{\frac{1}{2}}^{\frac{7}{8}} 2(y'(s)^2)ds \\ &\leq -\int_{\frac{3}{4}}^{\frac{7}{8}} 2(y'(s)^2)ds \\ &\leq -\frac{1}{4}y'(\tfrac{3}{4})^2, \end{aligned}$$

so $y'(\frac{3}{4})^2$ is also bounded as $k_1 \to -\infty$. Again using the inequality $y''(t) \geq 0$, we find that y' is bounded on $[\frac{1}{2}, \frac{3}{4}]$. This implies that $y - k_1$ is bounded on $[\frac{1}{2}, \frac{3}{4}]$, from which we find that as $k_1 \to -\infty$, e^{-2y} is getting arbitrarily large on $[\frac{1}{2}, \frac{3}{4}]$, whence the second equation of (4.12) implies that $y'(\frac{3}{4})$ is getting arbitrarily large, a contradiction.

Case 3: $y(1)$ is increasing as k_1 increases and $k^* > -\infty$. In this case, we claim that a solution of (4.12) subject to (4.14) exists for $k_1 = k^*$. To see this, take a sequence of $k_1 > k^*$ such that $k_1 \to k^*$. By our assumption on $y(1)$ and the monotonicity of y, we know that $|y(t)|$ is bounded on $[0,1]$, say by $R > 0$. Therefore, (4.15) implies that $0 \leq y'' \leq e^{2R} + \sqrt{2}\,(y')^2 = \phi(|y'|)$, where $\phi : [0, \infty)$ satisfies

$\int_0^\infty \frac{s}{\phi(s)} ds = \infty$. Lemma 5.1 in Chapter 12 of [35] then implies that there exists an $M > 0$ such that $|y'| < M$ on $[0,1]$. These estimates on y and y' alongside (4.12) imply that y is bounded in $C^3[0,1]$ and v is bounded in $C^2[0,1]$. The Arzela-Ascoli Theorem then implies that we have a convergent subsequence of (y,ξ) in $C^2[0,1] \times C^1[0,1]$. The limit is clearly going to be a solution of (4.12) subject to (4.14) with $k_1 = k^*$.

Since a solution (v^*, y^*) of (4.12) with (4.14) exists for $k_1 = k^*$, we can use basic perturbation arguments to prove existence of solutions to (4.12) and (4.14) for some values of $k_1 < k^*$, which contradicts the definition of k^*. To find these solutions, we aim to find v and y so that

$$\begin{aligned}
&(v - v^*)' = 2((y^*)' + y')((y^*)' - y'),\\
&(y - y^*)'' = \beta(e^{-2y} - e^{-2y^*}) - vy' + v^*(y^*)',\\
&v(\tfrac{1}{2}) = 0, \qquad y'(\tfrac{1}{2}) = 0, \qquad y(\tfrac{1}{2}) = k_1.
\end{aligned}$$

By writing $V = v - v^*$ and $Y = y - y^*$, we find

$$\begin{aligned}
&V' = -4(y^*)'Y' - 2(Y')^2,\\
&Y'' = \beta e^{-2y^*}(e^{-2Y} - 1) - VY' - v^*Y' - V(y^*)',\\
&V(\tfrac{1}{2}) = 0, \qquad Y'(\tfrac{1}{2}) = 0, \qquad Y(\tfrac{1}{2}) = k_1 - k^*.
\end{aligned} \tag{4.17}$$

As $k_1 \to k^*$, the values for $V(\frac{1}{2})$, $Y(\frac{1}{2})$ and $Y'(\frac{1}{2})$ approach a stationary solution of (4.17), so the solution of the initial value problem (4.17) can be made to exist on the entirety of $[\frac{1}{2}, 1]$ for k_1 close to k^*. □

Parts of the following sole-authored publications appear in Chapter 5:

1. [16] **T. Buttsworth**, The Dirichlet problem for Einstein metrics on cohomogeneity one manifolds, *Ann. Glob. Anal. Geom.*, 54, 1, 2018.

2. [17] **T. Buttsworth**, Cohomogeneity-one quasi-Einstein metrics, *J. Math. Anal. Appl.*, 470, 1, 2019.

Chapter 5

The Einstein and quasi-Einstein equations with $h = 0$

In the proofs of Theorems 2.2.3 and 2.2.4 we used the fact that the problems of solving (4.1) and (4.7) could be interpreted as finding fixed points of some $H : [0,1] \times X \to X$, where X is a Banach space. We relied on Lemmas 4.2.2 and 4.3.2 respectively, which suggested we could find a bounded, open and convex $\Omega \subset X$ such that $0 \notin (I - H(0,\cdot))(\partial\Omega)$ and $deg(I - H(0,\cdot),\Omega,0) \neq 0$. We present the proofs of these lemmas in this chapter.

5.1 Proof of Lemma 4.2.2: The Einstein equation with $h = 0$

We introduce the $d \times d$ diagonal matrix

$$L = diag\Big(\underbrace{\frac{f_1'}{f_1},\cdots,\frac{f_1'}{f_1}}_{d_1 \text{ times}},\cdots,\underbrace{\frac{f_n'}{f_n},\cdots,\frac{f_n'}{f_n}}_{d_n \text{ times}}\Big).$$

If $h = 0$, (4.1) becomes

$$tr(L)' = -tr(L^2) - \lambda, \tag{5.1}$$

$$L' = -tr(L)L - \lambda I, \tag{5.2}$$

$$\int_0^1 L = E. \tag{5.3}$$

Here, E is an $d \times d$ diagonal matrix with real entries D_i depending on the Dirichlet conditions of (4.1), I is the $d \times d$ identity matrix, and we aim to find a real number λ and a one-parameter family of diagonal matrices $L(r)$ that are smooth in r, and satisfy equations (5.1), (5.2) and (5.3). The proof of Lemma 4.2.2 follows from properties of solutions to this equivalent problem, which we discuss in this section through several lemmas.

Lemma 5.1.1. *If $(d-1)\lambda = tr(L(r)^2) - tr(L(r))^2$, for some $r \in [0,1]$, then the same holds for all r, provided equation* (5.2) *holds.*

Proof. This follows from the computation (4.2) in Lemma 4.2.1 with $h=0$. □

By combining equations (5.1) and (5.2), it is clear that solving equations (5.1), (5.2) and (5.3) is equivalent to solving equations (5.2), (5.3) and $(d-1)\lambda = tr(L(r))^2 - tr(L(r)^2)$. Then by Lemma 5.1.1, it suffices to solve (5.2), (5.3) and

$$(d-1)\lambda = tr(L(0))^2 - tr(L(0)^2). \tag{5.4}$$

We will now explore the solutions of this equivalent problem.

Lemma 5.1.2. *Let λ, D_i be constants and let $D=\sum_{i=1}^{d} D_i$. There exists a solution L of* (5.2) *and* (5.3) *if and only if $\lambda < \frac{\pi^2}{d}$. The solution is unique.*

Proof. Notice that whenever (5.2) and (5.3) hold, we must have

$$tr(L') = -tr(L)^2 - d\lambda \tag{5.5}$$

and

$$\int_0^1 tr(L) = D. \tag{5.6}$$

We consider separately the cases that λ is negative, positive and 0. In each case, we show that unless $\lambda \geq \frac{\pi^2}{d}$, we can uniquely solve (5.5) and (5.6). We then use the expression for $tr(L)$ to uniquely solve (5.2) and (5.3). In the case that λ is positive, we show that we need $\lambda < \frac{\pi^2}{d}$ for a solution to exist. This will complete the proof.

First Case. We assume $\lambda < 0$, and set $\mu = -\lambda > 0$. The general solution of (5.5) is given by

$$tr(L) = \sqrt{d\mu}\, \frac{e^{2\sqrt{d\mu}\, r} - C}{e^{2\sqrt{d\mu}\, r} + C} \tag{5.7}$$

for some constant C, or $tr(L) = -\sqrt{d\mu}$. We see that $tr(L) = -\sqrt{d\mu}$ satisfies (5.6) if and only if $D = -\sqrt{d\mu}$. Otherwise, $tr(L)$ must be given by (5.7), and we note that for this expression to be defined and continuous in r on $[0,1]$, we need $C \notin [-e^{2\sqrt{d\mu}}, -1]$. Furthermore, (5.6) holds if and only if the constant C is chosen so that

$$\begin{aligned} D &= \int_0^1 tr(L) \\ &= \left(\ln(|C+e^{2\sqrt{d\mu}\, r}|) - \sqrt{d\mu}\, r\right)\Big|_{r=0}^{r=1} \\ &= \ln(|C+e^{2\sqrt{d\mu}}|) - \sqrt{d\mu} - \ln(|C+1|). \end{aligned}$$

This is equivalent to

$$\left|\frac{C+e^{2\sqrt{d\mu}}}{C+1}\right| = e^{D+\sqrt{d\mu}}. \tag{5.8}$$

Since $C \notin [-e^{2\sqrt{d\mu}}, -1]$, we notice that $\left|\frac{C+e^{2\sqrt{d\mu}}}{C+1}\right| = \frac{C+e^{2\sqrt{d\mu}}}{C+1}$, and since $D + \sqrt{d\mu} \neq 0$, we can uniquely solve (5.8) for C with

$$C = \frac{e^{2\sqrt{d\mu}} - e^{D+\sqrt{d\mu}}}{e^{D+\sqrt{d\mu}} - 1},$$

and we note that $C \notin [-e^{2\sqrt{d\mu}}, -1]$ does indeed hold. Now that we have solved (5.5) and (5.6), we will solve (5.2) and (5.3). From (5.2) we find that

$$\begin{aligned} L_i &= \frac{\mu(e^{2\sqrt{d\mu r}} - C)}{\sqrt{d\mu}\,(C + e^{2\sqrt{d\mu}\,r})} + \frac{c_i\sqrt{d\mu}\,e^{\sqrt{d\mu}\,r}}{C + e^{2\sqrt{d\mu}\,r}} \\ &= \frac{1}{d}tr(L) + \frac{c_i\sqrt{d\mu}\,e^{\sqrt{d\mu}\,r}}{C + e^{2\sqrt{d\mu}\,r}} \end{aligned}$$

for some constants c_i, provided $tr(L)$ is not given by $tr(L) = -\sqrt{d\mu}$. The constants c_i can be found from (5.3) after integrating, and we see

$$D_i - \frac{D}{d} = c_i \int_0^1 \frac{\sqrt{d\mu}\,e^{\sqrt{d\mu}\,r}}{C + e^{2\sqrt{d\mu}\,r}}. \tag{5.9}$$

On the other hand, if $tr(L)$ is given by $tr(L) = -\sqrt{d\mu}$, then the solution of (5.2) is given by $L_i = \frac{1}{d}tr(L) + c_i\sqrt{d\mu}\,e^{\sqrt{d\mu}\,r}$ and the constants c_i are chosen so that

$$D_i - \frac{D}{d} = c_i \int_0^1 \sqrt{d\mu}\,e^{\sqrt{d\mu}\,r}. \tag{5.10}$$

Second Case. Now we assume $\lambda > 0$, in which case (5.5) implies that

$$tr(L) = -\sqrt{d\lambda}\,\tan(C + \sqrt{d\lambda}\,r), \tag{5.11}$$

for some constant C. For $tr(L)$ to be defined on $[0,1]$, we require that $\cos(C + \sqrt{d\lambda}\,r)$ does not change sign. This tells us that we need $\sqrt{d\lambda} < \pi$, which we assume is the case from now on. We can also add π to C if necessary to ensure that $\cos(C + \sqrt{d\lambda}\,r)$ is positive for all $r \in [0,1]$, because such a change in C does not change the value of $tr(L)$.

Equation (5.6) then implies that

$$D = \ln\cos(C + \sqrt{d\lambda}) - \ln\cos(C),$$

so we can see that C is chosen such that

$$\begin{aligned} e^D &= \frac{\cos(\sqrt{d\lambda} + C)}{\cos(C)} \\ &= \cos(\sqrt{d\lambda}) - \tan(C)\sin(\sqrt{d\lambda}). \end{aligned}$$

Rearranging, we see that

$$C = \arctan\left(\frac{\cos(\sqrt{d\lambda}) - e^D}{\sin(\sqrt{d\lambda})}\right) \in [-\frac{\pi}{2}, \frac{\pi}{2}],$$

and it is straightforward to show that given this choice of C, $\cos(C+\sqrt{d\lambda}\, r)$ is indeed positive for $r \in [0,1]$. Using (5.11), we can now solve equation (5.2) to find that

$$L_i = c_i \sec(C+\sqrt{d\lambda}\, r) + \frac{1}{d} tr(L)$$

for some constants c_i. Equation (5.3) then implies that the c_i constants must be chosen so that

$$D_i - \frac{D}{d} = \frac{c_i}{\sqrt{d\lambda}} \ln(\tan(C+\sqrt{d\lambda}\, r) + \sec(C+\sqrt{d\lambda}\, r))|_{r=0}^{r=1},$$

and we note that the input of the logarithm is indeed positive.

Third Case. Finally, if $\lambda = 0$, we split into two further cases: $D \neq 0$ and $D = 0$. If $D \neq 0$, (5.5) and (5.6) imply that $tr(L) = \frac{1}{C+r}$ for some constant $C \notin [-1,0]$ chosen so that $\ln\left(\frac{C+1}{C}\right) = D$. Rearranging gives $\frac{1}{e^D-1} = C$ which makes sense as $D \neq 0$. Then (5.2) implies that $L_i = \frac{tr(L)}{d} + \frac{c_i}{C+r}$ and (5.3) implies that the c_i terms are uniquely chosen so that $D_i - \frac{D}{d} = c_i D$.

If $D = 0$, then (5.5) combined with (5.6) implies that $tr(L) = 0$. Equations (5.2) and (5.3) then imply that $L_i = D_i$. □

For a given $\lambda < \frac{\pi^2}{d}$, Lemma 5.1.2 implies that a solution of (5.2) and (5.3) exists and is unique. As found in the proof of Lemma 5.1.2, the solution is

$$L_i(r) = \begin{cases} \frac{\mu(e^{2\sqrt{d\mu}\, r} - C_-)}{\sqrt{d\mu}\,(C_- + e^{2\sqrt{d\mu}\, r})} + \frac{c_{-i}\sqrt{d\mu}\, e^{\sqrt{d\mu}\, r}}{C_- + e^{2\sqrt{d\mu}\, r}} & \text{if } -\lambda = \mu > 0 \text{ and } \sqrt{d\mu} \neq -D, \\ \frac{-\sqrt{d\mu}}{d} + \tilde{c}_{-i}\sqrt{d\mu}\, e^{\sqrt{d\mu}\, r} & \text{if } -\lambda = \mu > 0 \text{ and } \sqrt{d\mu} = -D, \\ \frac{1}{d(C_0+r)} + \frac{c_{0i}}{C_0+r} & \text{if } \lambda = 0 \text{ and } D \neq 0, \\ D_i & \text{if } \lambda = 0 \text{ and } D = 0, \\ c_{+i}\sec(C_+ + \sqrt{d\lambda}\, r) - \frac{\sqrt{d\lambda}}{d}\tan(C_+ + \sqrt{d\lambda}\, r) & \text{if } 0 < \lambda < \frac{\pi^2}{d}, \end{cases} \tag{5.12}$$

where

$$C_- = \frac{e^{2\sqrt{d\mu}} - e^{(D+\sqrt{d\mu})}}{e^{(D+\sqrt{d\mu})} - 1},\ c_{-i} = \frac{D_i - \frac{D}{d}}{\int_0^1 \frac{\sqrt{d\mu}\, e^{\sqrt{d\mu}\, r}}{C_- + e^{2\sqrt{d\mu}\, r}}},\ \tilde{c}_{-i} = \frac{D_i - \frac{D}{d}}{\int_0^1 \sqrt{d\mu}\, e^{\sqrt{d\mu}\, r}},$$

$$C_0 = \frac{1}{e^D - 1},\ c_{0i} = \frac{D_i - \frac{D}{d}}{D},$$

$$C_+ = \arctan\left(\frac{\cos(\sqrt{d\lambda}) - e^D}{\sin(\sqrt{d\lambda})}\right),\ c_{+i} = \frac{\sqrt{d\lambda}\,(D_i - \frac{D}{d})}{\ln(\tan(C_+ + \sqrt{d\lambda}\, r) + \sec(C_+ + \sqrt{d\lambda}\, r))|_{r=0}^{r=1}}.$$

We will put these solutions of (5.2) and (5.3) into (5.4) to find the value of λ. The following lemma imposes some initial constraints on the possible values of λ.

Lemma 5.1.3. *If the unique solution of* (5.2) *and* (5.3) *satisfies* (5.4)*, then* $\lambda \in [\frac{-D^2}{d}, \frac{\pi^2}{d})$.

Proof. We already know that $\lambda < \frac{\pi^2}{d}$ for a solution of (5.2) to even exist. Now, if $\lambda = -\mu < -\frac{D^2}{d}$, then $C_- > 0$. Substituting our solution of (5.2) and (5.3) into (5.4) yields $(1-d)\mu = d\mu\frac{(1-C_-)^2}{(1+C_-)^2}(\frac{1}{d} - 1) + \frac{\sum_{i=1}^{d} d\mu c_{-i}^2}{(C_-+1)^2}$ which can be rearranged to

$$4(\frac{1}{d}-1)C_- = \sum_{i=1}^{d} c_{-i}^2. \tag{5.13}$$

This is a contradiction since $C_- > 0$. □

Now solving (5.4) for $\lambda \in [-\frac{D^2}{d}, \frac{\pi^2}{d})$ will involve the function $m : [-\frac{D^2}{d}, \frac{\pi^2}{d}) \to \mathbb{R}^+$ given by

$$m(\lambda) = \begin{cases} 0 & \text{if } 0 < d\mu = -d\lambda = D^2, \\ \left| \ln\left(\frac{(\frac{e\sqrt{d\mu}}{\sqrt{-C_-}}+1)(-\frac{1}{\sqrt{-C_-}}+1)}{(1-\frac{e\sqrt{d\mu}}{\sqrt{-C_-}})(1+\frac{1}{\sqrt{-C_-}})} \right) \right| & \text{if } 0 < d\mu = -d\lambda < D^2, \\ |D| & \text{if } \lambda = 0, \\ \left| \ln \frac{\tan(C_+ + \sqrt{d\lambda}\,) + \sec(C_+ + \sqrt{d\lambda}\,)}{\tan(C_+) + \sec(C_+)} \right| & \text{if } 0 < d\lambda < \pi^2, \end{cases}$$

where we treat C_- and C_+ as functions of λ. The importance of m is demonstrated with the following lemma.

Lemma 5.1.4. *The solution of* (5.2) *and* (5.3) *satisfies* (5.4) *if and only if*

$$m(\lambda) = \sqrt{\frac{d}{d-1}\sum_{i=1}^{d}\left(D_i - \frac{D}{d}\right)^2}. \tag{5.14}$$

Proof. First, if $-\frac{D^2}{d} = \lambda < 0$, then $\sqrt{d\mu} = \pm D$, and $L_i(r)$ is given by the second line of (5.12), or by the first line with $C_- = 0$. Substituting these expressions into (5.4) gives $\sum_{i=1}^{d} \tilde{c}_{-i}^2 = 0$ or $\sum_{i=1}^{d} c_{-i}^2 = 0$. This is equivalent to $\sum_{i=1}^{d}(D_i - \frac{D}{d})^2 = 0$, which is equivalent to $m(\lambda) = 0$, as required.

Next, assume that $-\frac{D^2}{d} < \lambda < 0$. Then $C_- < 0$, so like in the proof of Lemma 5.1.3 we deduce that (5.4) is equivalent to

$$4\left(\frac{1}{d}-1\right)C_- = \sum_{i=1}^{d} c_{-i}^2. \tag{5.15}$$

By substituting our definitions of C_- and c_{-i} into (5.15), explicitly evaluating the integral in the definition of c_{-i} and rearranging for $\sum_{i=1}^{d}(D_i - \frac{D}{d})^2$, we find that (5.15) is equivalent to (5.14).

If $\lambda = 0$ and $D = 0$, then substituting our solution of (5.2) and (5.3) into (5.4) gives $\sum_{i=1}^{d}(D_i - \frac{D}{d})^2 = 0$, as required. If $D \neq 0$ and $\lambda = 0$, then the same process gives $\frac{1}{C_0^2}(\frac{1}{d} - 1 + \sum_{i=1}^{d} c_{0i}^2) = 0$. Since $C_0 \neq 0$, we can multiply this equation by C_0^2 and use our expression for c_{0i} to find that (5.4) is equivalent to (5.14).

If $\lambda > 0$, then (5.4) is equivalent to

$$(d-1)\lambda = (1-d)\lambda\tan^2(C_+) + \sec^2(C_+)\sum_{i=1}^{d} c_{+i}^2. \tag{5.16}$$

After using our definitions of C_+ and c_{+i} and rearranging we see that (5.16) is equivalent to (5.14). □

The combination of Lemmas 5.1.2, 5.1.3 and 5.1.4 demonstrates that uniquely solving (5.1), (5.2) and (5.3) is equivalent to uniquely solving the equation $m(\lambda) = \sqrt{\frac{d}{d-1}\sum_{i=1}^{d}(D_i - \frac{D}{d})^2}$ for λ. The following two lemmas show us that this is indeed possible.

Lemma 5.1.5. *The function* $m : [\frac{-D^2}{d}, \frac{\pi^2}{d}) \to \mathbb{R}^+$ *is surjective.*

Proof. We demonstrate that the image of m is $\mathbb{R}^+$ by computing some limits. First we check limits of m on $(-\frac{D^2}{d}, 0)$, assuming $D \neq 0$. As $\mu \to 0$, the quantity $\frac{1-\frac{1}{\sqrt{-C_-}}}{1-\frac{e^{\sqrt{d\mu}}}{\sqrt{-C_-}}}$ converges to e^{-D} by L'Hôpital's rule. Therefore,

$$\lim_{\lambda\to 0^-} m(\lambda) = |D|. \tag{5.17}$$

As $\sqrt{d\mu} \to |D|$, C_- goes to 0 or $-\infty$ (depending on the sign of $D \neq 0$), so

$$\lim_{\lambda\to -\frac{D^2}{d}} m(\lambda) = 0. \tag{5.18}$$

Now for the limits on $(0, \frac{\pi^2}{d})$. We see that

$$\frac{\tan(C_+ + \sqrt{d\lambda}\,) + \sec(C_+ + \sqrt{d\lambda}\,)}{\tan(C_+) + \sec(C_+)} = \frac{\cos(C_+)}{\cos(C_+ + \sqrt{d\lambda}\,)} \frac{\sin(C_+ + \sqrt{d\lambda}\,) + 1}{\sin(C_+) + 1}$$

and $\frac{\cos(C_+)}{\cos(C_+ + \sqrt{d\mu}\,)} = e^{-D}$ by the identity $\frac{\cos(x+y)}{\cos(x)} = \cos(y) - \tan(x)\sin(y)$. If $D \leq 0$, then C_+ stays away from $-\frac{\pi}{2}$ as $\lambda \to 0^+$, so $\lim_{\lambda\to 0^+} \frac{\sin(C_+ + \sqrt{d\lambda}\,)+1}{\sin(C_+)+1} = 1$. On the other hand, if $D > 0$, then $C_+ \to -\frac{\pi}{2}$ as $\lambda \to 0^+$, so by L'Hôpitals rule,

$$\lim_{\lambda\to 0^+} \frac{\sin(C_+ + \sqrt{d\lambda}\,) + 1}{\sin(C_+) + 1} = \lim_{\lambda\to 0^+} \frac{\cos(C_+ + \sqrt{d\lambda}\,)}{\cos(C_+)} \left(1 + \frac{\sin^2(0) + (\cos(0) - e^D)^2}{e^D - 1}\right) = e^{2D}.$$

In either case, we can see that

$$\lim_{\lambda\to 0^+} m(\lambda) = |D|. \tag{5.19}$$

It is clear that m is continuous on $(-\frac{D^2}{d}, 0)$ and $(0, \frac{\pi^2}{d})$. The three limits (5.17), (5.18) and (5.19) demonstrate that m is also continuous at 0 and $-\frac{D^2}{d}$, so m is in fact continuous on all of $[-\frac{D^2}{d}, \frac{\pi^2}{d})$.

To conclude the proof, notice that if $\sqrt{d\lambda} \to \pi$, then $C_+ \to -\frac{\pi}{2}$, so $\frac{\sin(C_+ + \sqrt{d\lambda}\,)+1}{\sin(C_+)+1}$ goes to ∞, and

$$\lim_{\lambda\to \frac{\pi^2}{d}} m(\lambda) = \infty.$$

Therefore, m is continuous, gets arbitrarily close to 0 and also gets arbitrarily large. The intermediate value theorem then implies that m achieves all values in $\mathbb{R}^+$. □

Lemma 5.1.6. *The function m is monotone increasing.*

Proof. Fix $D \in \mathbb{R}$. In Lemma 5.1.5 we established that m is continuous, $m(-\frac{D^2}{d}) = 0$ and $\lim_{\lambda \to \frac{\pi^2}{d}} m(\lambda) = \infty$. Therefore, if m is not monotone increasing, it is not injective. Therefore, we can find values of D_i satisfying $\sum_{i=1}^d D_i = D$ so that $m(\lambda) = \sqrt{\frac{d}{d-1}\sum_{i=1}^d \left(D_i - \frac{D}{d}\right)^2}$ has multiple solutions, i.e., for some numbers D_i and D, the solution of (5.1), (5.2) and (5.3) is non-unique. We conclude the proof by showing that this is not the case.

Consider the functions $M_i = L_i - \frac{tr(L)}{d}$. These functions solve the equation

$$M_i' = -tr(L)M_i,$$

subject to the integral condition $\int_0^1 M_i(r)dr = D_i - \frac{D}{d}$, so

$$M_i(r) = (D_i - \frac{D}{d})\frac{e^{p(r)}}{\int_0^1 e^{p(t)}dt},$$

where $p(r) = -\int_0^r tr(L(t))dt$. It follows from (5.1) and the fact that $(d-1)\lambda = tr(L^2) - tr(L)^2$ that p solves the equation

$$\begin{aligned}
-p'' &= -tr(L^2) - \lambda \\
&= -tr(L^2) + \frac{1}{d-1}(tr(L)^2 - tr(L^2)) \\
&= \frac{(p')^2}{d-1} - (1 + \frac{1}{d-1})\sum_{i=1}^d \left(M_i - \frac{p'}{d}\right)^2 \\
&= \frac{(p')^2}{d-1} - \frac{d}{d-1}\sum_{i=1}^d \left((D_i - \frac{D}{d})\frac{e^{p(r)}}{\int_0^1 e^{p(t)}dt} - \frac{p'}{d}\right)^2,
\end{aligned}$$

which simplifies to

$$p'' = \left(\frac{e^p}{\int_0^1 e^p}\right)^2 \frac{d}{d-1}\sum_{i=1}^d \left(D_i - \frac{D}{d}\right)^2. \tag{5.20}$$

We also note that p satisfies the Dirichlet conditions

$$p(0) = 0, \qquad p(1) = -D. \tag{5.21}$$

We claim that solutions of (5.20) subject to (5.21) are unique. Indeed, if $\sum_{i=1}^d \left(D_i - \frac{D}{d}\right)^2 = 0$, the result is obvious. Otherwise, take another solution $p + \tilde{p}$, so that

$$\begin{aligned}
&\tilde{p}'' = \frac{d}{d-1}\sum_{i=1}^d \left(D_i - \frac{D}{d}\right)^2 \left(\frac{e^{p+\tilde{p}}}{\int_0^1 e^{p+\tilde{p}}} - \frac{e^p}{\int_0^1 e^p}\right)\left(\frac{e^{p+\tilde{p}}}{\int_0^1 e^{p+\tilde{p}}} + \frac{e^p}{\int_0^1 e^p}\right), \\
&\tilde{p}(0) = 0, \qquad \tilde{p}(1) = 0.
\end{aligned}$$

If $\tilde{p}$ achieves a positive maximum at a point t, then

$$\tilde{p}''(t) > \frac{d}{d-1}\sum_{i=1}^d \left(D_i - \frac{D}{d}\right)^2 \left(\frac{e^{p(t)+\tilde{p}(t)}}{e^{\tilde{p}(t)}\int_0^1 e^p} - \frac{e^{p(t)}}{\int_0^1 e^p}\right)\left(\frac{e^{p(t)+\tilde{p}(t)}}{\int_0^1 e^{p+\tilde{p}}} + \frac{e^{p(t)}}{\int_0^1 e^p}\right) = 0,$$

which is a contradiction. We similarly find a contradiction if $\tilde{p}$ achieves a negative minimum, so p is uniquely determined. It follows that L_i is also uniquely determined, as is λ. □

Now that we have existence and uniqueness of solutions of (5.1), (5.2) and (5.3), we demonstrate that these solutions behave well under changes of values of D_i.

Lemma 5.1.7. *Suppose that $|D_i| \le R$ for each $i=1,\cdots,d$. Then there exists $R'>0$ depending only on R and d such that the solution (λ, L_i) of (5.1), (5.2) and (5.3) satisfies $|\lambda|<R'$ and $|L_i|<R'$ for each $i=1,\cdots,d$.*

Proof. Assume to the contrary that no such $R'>0$ exists. Then there exists an unbounded sequence of solutions $(\lambda^{(j)}, L_i^{(j)})$ to (5.1), (5.2) and (5.3) with $\left|D_i^{(j)}\right| \le R$ for each $i=1,\cdots,d$. Here, $j\in\mathbb{N}$ is used to distinguish the different elements of the sequence.

By taking a subsequence of $(\lambda^{(j)}, L_i^{(j)})$ if necessary, we can assume that the $\lambda^{(j)}$ terms are monotone increasing or decreasing. We already know that $\lambda^{(j)} \in [-\frac{(D^{(j)})^2}{d}, \frac{\pi^2}{d}] \subseteq [-dR^2, \frac{\pi^2}{d}]$, so the $\lambda^{(j)}$ terms are bounded, hence convergent. We claim that $\lambda_{(j)}$ converges to $\frac{\pi^2}{d}$ from below. If this were not the case, then there would be some $K<\frac{\pi^2}{d}$ such that $-dR^2 \le \lambda^{(j)} \le K$. Taking the trace of (5.2) implies that $tr(L^{(j)})' = -tr(L^{(j)})^2 - d\lambda^{(j)}$ subject to $\int_0^1 tr(L^{(j)}) = D^{(j)}$. Similarly to the proof of Lemma 5.1.2, we deduce that the solution of this equation is

$$tr(L^{(j)})(r) = \begin{cases} \frac{\sqrt{d\mu^{(j)}}\,(e^{2\sqrt{d\mu^{(j)}}\,r} - C_-^{(j)})}{(C_-^{(j)} + e^{2\sqrt{d\mu^{(j)}}\,r})} & \text{if } -\lambda^{(j)} = \mu^{(j)} > 0 \text{ and } \sqrt{d\mu^{(j)}} \ne -D^{(j)}, \\ -\sqrt{d\mu^{(j)}} & \text{if } -\lambda^{(j)} = \mu^{(j)} > 0 \text{ and } \sqrt{d\mu^{(j)}} = -D^{(j)}, \\ \frac{1}{(C_0^{(j)}+r)} & \text{if } \lambda^{(j)} = 0 \text{ and } D^{(j)} \ne 0, \\ 0 & \text{if } \lambda^{(j)} = 0 \text{ and } D^{(j)} = 0, \\ -\sqrt{d\lambda^{(j)}}\,\tan(C_+^{(j)} + \sqrt{d\lambda^{(j)}}\,r) & \text{if } 0 < \lambda^{(j)} < \frac{\pi^2}{d}, \end{cases}$$

where

$$\begin{aligned} C_-^{(j)} &= \frac{e^{2\sqrt{d\mu^{(j)}}} - e^{(D^j + \sqrt{d\mu^{(j)}})}}{e^{(D^{(j)} + \sqrt{d\mu^{(j)}})} - 1}, \\ C_0^{(j)} &= \frac{1}{e^{D^{(j)}} - 1}, \\ C_+^{(j)} &= \arctan\left(\frac{\cos(\sqrt{d\lambda^{(j)}}) - e^{D^{(j)}}}{\sin(\sqrt{d\lambda^{(j)}})}\right). \end{aligned}$$

Since $-dR^2 \le \lambda^{(j)} \le K < \frac{\pi^2}{d}$ and $\left|D^{(j)}\right| \le dR$, it is straightforward to show that $tr(L^{(j)})$ is bounded independently of j. Now if we treat $tr(L^{(j)})$ as a given function, we can think of (5.2) coupled with (5.3) as a linear equation for $L_i^{(j)}$. This equation is easily solved in terms of $tr(L^{(j)})$, and the bound on $tr(L^{(j)})$ then implies that $L_i^{(j)}$ itself is bounded independently of j for each $i=1,\cdots,d$. However, we now have bounds on both $\lambda^{(j)}$ and $L_i^{(j)}$, which contradicts the assumption that $(\lambda^{(j)}, L_i^{(j)})$ is unbounded.

We now know that $\lambda^{(j)} \to \frac{\pi^2}{d}$ from below. However, in this case, $C_+^{(j)} \to -\frac{\pi}{2}$, which implies that $m(\lambda^{(j)}) \to \infty$. This is a contraction because $m(\lambda^{(j)})$ must coincide with $\sqrt{\frac{d}{d-1}\sum_{i=1}^d (D_i^{(j)} - \frac{D^{(j)}}{d})^2}$, which is bounded. ☐

We are now in a position to conclude our examination of the Einstein equation.

Proof of Lemma 4.2.2. Choose continuous functions $A_i : [0,1] \to \mathbb{R}^+$ and $B_i : [0,1] \to \mathbb{R}^+$ so that $A_i(0) = B_i(0)$, and $A_i(1) = a_i$, $B_i(1) = b_i$. Consider the problem of solving

$$\begin{aligned} \sum_{i=1}^{n} d_i \left(-\frac{f_i'}{f_i} \sum_{k=1}^{n} d_k \frac{f_k'}{f_k} + \frac{f_i'^2}{f_i^2} \right) |_{r=0} &= (d-1)\lambda, \\ -\frac{f_i'}{f_i} \sum_{k=1}^{n} d_k \frac{f_k'}{f_k} + \frac{f_i'^2}{f_i^2} - \frac{f_i''}{f_i} &= \lambda \text{ for } i = 1, \cdots, n, \\ f_i(0) = A_i(p), \qquad f_i(1) &= B_i(p) \text{ for } i = 1, \cdots, n, \end{aligned} \tag{5.22}$$

for some $p \in [0,1]$. Proceeding as we did in Section 4.2, we can find $E : [0,1] \times \mathbb{R} \times C^1([0,1] : (\mathbb{R}^n)^+) \to \mathbb{R} \times C^1([0,1] : \mathbb{R}^n)$ so that fixed points of $E(p,\cdot)$ are precisely solutions of (5.22). This function is still compact, and furthermore, Lemma 5.1.7 implies that it is possible to choose $\Omega \subset \mathbb{R} \times C^1([0,1] : (\mathbb{R}^n)^+)$ so that Ω contains all of the fixed points of $E(p,\cdot)$ for an arbitrary $p \in [0,1]$. The only fixed point of $E(0,\cdot)$ is $f_i = A_i(0) = B_i(0)$ and $\lambda = 0$, and it is clear that the linearisation of $E(0,\cdot)$ at this solution is trivial, so we find that $deg(I - E(0,\cdot), \Omega, 0) = 1$. Property (ii) of Theorem 4.1.1 then implies that $deg(I - H(0,\cdot), \Omega, 0) = deg(I - E(1,\cdot), \Omega, 0) = 1$ as well. □

5.2 Proof of Lemma 4.3.2: The quasi-Einstein equation with $h = 0$

We again use the diagonal matrix

$$L = diag\Big(\underbrace{\frac{f_1'}{f_1}, \cdots, \frac{f_1'}{f_1}}_{d_1 \text{ times}}, \cdots, \underbrace{\frac{f_n'}{f_n}, \cdots, \frac{f_n'}{f_n}}_{d_n \text{ times}} \Big),$$

and also define $v = \xi'$. We then see that solving $H(0,\xi,f) = (\xi,f)$ is equivalent to

$$\begin{aligned} v' &= -tr(L^2) - m(tr(L) - v)^2, \\ L' &= -vL, \end{aligned} \tag{5.23}$$

subject to

$$\int_0^1 L = D, \qquad \int_0^1 v = w, \tag{5.24}$$

for some diagonal matrix D and a real number w, both depending on the Dirichlet conditions. In particular,

$$D = diag\Big(\underbrace{\ln(b_1) - \ln(a_1), \cdots, \ln(b_1) - \ln(a_1)}_{d_1 \text{ times}}, \cdots, \underbrace{\ln(b_n) - \ln(a_n), \cdots, \ln(b_n) - \ln(a_n)}_{d_n \text{ times}} \Big),$$

and $w = \xi(1) - \xi(0)$. The following lemma is a compactness result for solutions of (5.23) and (5.24), in that it demonstrates that a sequence of solutions has a convergent subsequence in a certain weak sense.

Lemma 5.2.1. *Suppose that there is a sequence of solutions* $(L^{(k)}, v^{(k)}, D^{(k)}, w^{(k)})$ *to* (5.23) *and* (5.24) *on* $[0,1]$ *with* $\left|D^{(k)}\right| + \left|w^{(k)}\right| < R$ *for some* $R > 0$*. Then there exists a solution* $(\bar{L}, \bar{v})$ *to* (5.23) *defined on* $(0,1)$ *such that for each* $0 < a < \frac{1}{2} < b < 1$*, there exists a subsequence of* $(L^{(k)}, v^{(k)})$ *converging to* $(\bar{L}, \bar{v})$ *in* $C^0[a,b]$*.*

Proof. By taking a first subsequence of $(L^{(k)}, v^{(k)})$, we can assume that for each i, $L_i^{(k)}(\frac{1}{2})$ and $v^{(k)}(\frac{1}{2})$ are monotone in k. Fix $0 < a < \frac{1}{2} < b < 1$. The proof relies on showing that $(L^{(k)}, v^{(k)})$ is bounded in $C^0[a,b]$ independently of $k \in \mathbb{N}$. If we can do this, since $(L^{(k)}, v^{(k)})$ solves (5.23), we also find that $(L^{(k)}, v^{(k)})$ is bounded in $C^2[a,b]$. Therefore, the Arzela-Ascoli Theorem implies that there is a subsequence of $(L^{(k)}, v^{(k)})$ converging to $(\bar{L}, \bar{v})$ in $C^1[a,b]$, so $(\bar{L}, \bar{v})$ is a solution of (5.23). Since $L_i^{(k)}(\frac{1}{2})$ and $v^{(k)}(\frac{1}{2})$ are monotone in k, $\bar{L}(\frac{1}{2})$ and $\bar{v}(\frac{1}{2})$ are independent of a and b, and by uniqueness of solutions to ODEs, $\bar{L}$ and $\bar{v}$ are themselves independent of a and b, and can be extended to a solution on $(0,1)$ as required. We now drop reference to the superscript k to simplify notation.

To find the required bounds on (L, v), first suppose that there exists an i such that $\sup_k ||L_i||_{C^0[a,b]} = \infty$. Note that the second equation of (5.23) implies that

$$(L_i^2)' = -2L_i^2 v, \tag{5.25}$$

from which we can see that L_i does not change sign on $[0,1]$. Also, from the first equation of (5.23), we see that $v' \leq 0$, so if L_i is non-zero, it can have at most one critical point. If this critical point exists, it minimises L_i^2 by (5.25). Therefore, if L_i is unbounded on $[a,b]$, then $L_i^2(a)$ is unbounded and $L_i^2 \geq L_i^2(a)$ on $[0,a]$, or $L_i^2(b)$ is unbounded and $L_i^2 \geq L_i^2(b)$ on $[b,1]$. In either case, we will eventually get a contradiction with (5.24).

Now suppose that v is unbounded. Since $v' \leq 0$, we can find a subsequence such that $v(a) \to \infty$ or $v(b) \to -\infty$ monotonically in k. If $v(b) \to -\infty$, then since $v \leq v(b)$ on $[b,1]$, $(L_i^2)' \geq 2L_i^2(-v(b))$ on $[b,1]$, and in order for (5.24) to be satisfied, $L_i(b)$ must converge to 0 for each i. Similarly, if $v(a) \to \infty$, then $L_i(a) \to 0$ for each i. In each case, we claim that far enough along the sequence, v does not change sign on $[0,1]$. If it did, L_i^2 would be minimised where v is 0, so at this point, $L_i^2 \leq \min\{L_i(a)^2, L_i(b)^2\} \to 0$. We therefore have points $t^{(k)} \in [0,1]$ such that $(L^{(k)}, v^{(k)})(t^{(k)})$ is getting arbitrarily close to the $(0,0)$ critical point of (5.23), which contradicts the assumption that $(L^{(k)}, v^{(k)})$ is unbounded on $[0,1]$. Therefore, v does not change sign far enough along the sequence. However, since v does not change sign, $v(t)$ is monotone decreasing in t and we know that $v(a) \to \infty$ or $v(b) \to -\infty$, we have a contradiction with (5.24). □

We now use Lemma 5.2.1 to demonstrate that if we have a sequence of solutions to (5.23) and (5.24), then these solutions are bounded.

Lemma 5.2.2. *Suppose* $|D| + |w| < R$ *for some* $R > 0$*. Then there exists* $R' > 0$ *depending only on* R *and* d *so that any solution of* (5.23) *and* (5.24) *is bounded in the* C^0 *sense by* R'*.*

Proof. Assume to the contrary that no such bound exists. Then there exists a sequence of solutions $(L^{(k)}, v^{(k)}, D^{(k)}, v^{(k)})$ such that $(L^{(k)}, v^{(k)})$ are unbounded in the C^0 sense and $\left|D^{(k)}\right| + \left|w^{(k)}\right| < R$. We can assume that the C^0 norm of $(L^{(k)}, v^{(k)})$ is monotone increasing in k. Similarly to the proof of Lemma 5.2.1, we now drop reference to the superscript where convenient to do so.

In order to keep track of the blow-up, we will let $M = \sqrt{tr(L^2)+v^2}$, and note that there exists $s>0$ such that

$$|M'| \leq sM^2. \tag{5.26}$$

Now take a subsequence such that either v does not change sign for each element of the sequence, or v does change sign for each element of the sequence. If v does not change sign, then $\int_0^1 M \leq |w|+|D|$. However, since M is unbounded and satisfies (5.26), we find that $\int_0^1 M$ is also unbounded, which is a contradiction.

We can now assume that v changes sign for each element of our sequence. Let $(\bar{L},\bar{v})$ be the solution found by Lemma 5.2.1. We claim that this solution is unbounded. Indeed, if there were some R'' which bounds $(\bar{L},\bar{v})$, choose a and b close to 0 and 1 respectively so that a solution of (5.23) bounded by $R''+1$ on $[a,b]$ stays bounded by $R''+2$ on $[0,1]$. Now by Lemma 5.2.1, we have a subsequence which is convergent to $(\bar{L},\bar{v})$ in the $C^0[a,b]$ sense. Therefore, far enough along the subsequence, (L,v) will be bounded by $R''+1$ on $[a,b]$, and so must also be bounded by $R''+2$ on $[0,1]$. This contradicts the assumption that our sequence is unbounded in C^0.

We now know that $(\bar{L},\bar{v})$ is unbounded. The remainder of the proof involves showing that the nature of the unboundedness implies that far enough along our sequence, $L^{(k)}$ and $v^{(k)}$ cannot possibly satisfy (5.24).

Case 1: our solution $(\bar{L},\bar{v})$ is unbounded around 0. Letting $\bar{M}^2 = \bar{v}^2 + tr(\bar{L}^2)$, we have again $\bar{M}' \geq -s\bar{M}^2$. Since $\bar{v}$ is monotone decreasing and $\bar{L}_i$ is monotone on any interval where $\bar{v}$ does not change sign, we see that $\bar{M}$ blows up at 0, so we must have $\bar{M}(t) \geq \frac{1}{st}$. Also note that for small t, $\bar{v}$ has to become positive, otherwise neither $\bar{v}$ nor $\bar{L}_i$ can become unbounded around 0. Therefore, for small t,

$$\bar{M} \leq \sum_{i=1}^{d} |\bar{L}_i| + \bar{v}. \tag{5.27}$$

Choose i such that $|\bar{L}_i(t)|$ is maximised for all $t\in[0,1]$. Such an i must exist and is independent of $t\in[0,1]$ because of the second equation of (5.23).

Case 1a: $|\bar{L}_i| = 0$, so $\bar{L}=0$ on $[0,1]$. Since v changes sign, $tr(L^2)$ is minimised at the point where $v=0$. Now fix some $0<a<\frac{1}{2}<b<1$, and take a subsequence of (L,v) that converges uniformly to $(0,\bar{v})$ on $[a,b]$. Since L converges uniformly to 0 on $[a,b]$, the minimum value of $tr(L^2)$ on $[0,1]$ must also converge to 0. Then on this subsequence, we have points $t^{(k)}$ such that $v^{(k)}(t^{(k)})=0$ and $tr((L^{(k)})^2(t^{(k)})) \to 0$, so $M^{(k)}(t^{(k)}) \to 0$. The inequality (5.26) then implies that $M^{(k)} \to 0$ uniformly on $[0,1]$, which is a contradiction because M is unbounded.

Case 1b: $|\bar{L}_i| = \bar{L}_i > 0$. In this case, (5.27) implies that $\bar{L}_i' = -\bar{v}\bar{L}_i \leq -\bar{L}_i(\frac{1}{st} - d\bar{L}_i)$. The solution of the differential equation $x' = -x(\frac{1}{st} - dx)$ is $x(t) = \frac{s}{t(C - sd\ln(\frac{t}{s}))}$ for some $C\in\mathbb{R}$. If we choose C so that $x(\frac{1}{2}) = \bar{L}_i(\frac{1}{2})$, then for all $t\leq\frac{1}{2}$, $\bar{L}_i(t)\geq x(t)$. Therefore $\int_0^1 \bar{L}_i \geq \int_0^{\frac{1}{2}} \bar{L}_i = \infty$, which implies that $\int_a^b \bar{L}_i$ becomes arbitrarily large by having a and b close to 0 and 1 respectively. However, on $[a,b]$, we have a subsequence of (L,v) converging to $(\bar{L},\bar{v})$ in $C^0[a,b]$, and by taking another subsequence, we can assume that $L_i > 0$ for each element of the sequence. We find that

$$D_i = \int_0^1 L_i \geq \int_a^b L_i \to \int_a^b \bar{L}_i,$$

which is a contradiction.

Case 1c: $|\bar{L}_i| = -\bar{L}_i > 0$. This time, (5.27) implies that $\bar{L}_i' \geq -\bar{L}_i(\frac{1}{st} + d\bar{L}_i)$. Letting $L^* = -\bar{L}_i$, we find that $L^* > 0$ and $(L^*)' \leq -L^*(\frac{1}{st} - dL^*)$. This is the same inequality as in Case 1 b, so we proceed as before to find a contradiction.

Case 2: $(\bar{L}, \bar{v})$ is unbounded at 1. We do not need to treat this case because the transformation $L \to -L$, $v \to -v$ and $t \to 1-t$ leaves (5.23) invariant, but changes the signs of w and the diagonal components of D, and shifts the location of the unboundedness from 1 to 0. The situation is then treatable by Case 1. □

We can now prove the main result of this section.

Proof of Lemma 4.3.2. Choose continuous functions $A_i : [0,1] \to \mathbb{R}^+$ and $B_i : [0,1] \to \mathbb{R}^+$ so that $A_i(1) = a_i$, $B_i(1) = b_i$ and $A_i(0) = B_i(0)$. Also choose continuous functions $C_0 : [0,1] \to \mathbb{R}$ and $C_1 : [0,1] \to \mathbb{R}$ so that $C_0(0) = C_1(0) = 0$ and $C_0(1) = \sum_{i=1}^n d_i b_i - c_1$, $C_1(1) = \sum_{i=1}^n d_i b_i - c_1$ and consider the problem of solving

$$\begin{aligned}
\xi'' + \sum_{k=1}^n d_k \frac{f_k'^2}{f_k^2} + m(\sum_{k=1}^n d_k \frac{f_k'}{f_k} - \xi')^2 &= -h^2\lambda, \\
\frac{h^2\beta_i}{2f_i^2} + h^2 \sum_{k,l=1}^n \gamma_{ik}^l \frac{f_i^4 - 2f_k^4}{4f_i^2 f_k^2 f_l^2} - \xi' \sum_{k=1}^n d_k \frac{f_k'}{f_k} + \frac{f_i'^2}{f_i^2} - \frac{f_i''}{f_i} &= h^2\lambda \text{ for } i = 1, \cdots, n, \\
f_i(0) = A_i(p), \qquad f_i(1) &= B_i(p), \text{ for } i = 1, \cdots, n, \\
\xi(0) = C_0(p), \qquad \xi(1) &= C_1(p)
\end{aligned} \tag{5.28}$$

for some $p \in [0,1]$. Proceeding as we did in Section 4.2, we can find $E : [0,1] \times C^1([0,1] : \mathbb{R}) \times C^1([0,1] : (\mathbb{R}^n)^+) \to C^1([0,1] : \mathbb{R}) \times C^1([0,1] : \mathbb{R}^n)$ so that fixed points of $E(p,\cdot)$ are precisely solutions of (5.22). This function is still compact, and furthermore, Lemma 5.2.2 implies that it is possible to choose $\Omega \subset C^1([0,1] : \mathbb{R}) \times C^1([0,1] : (\mathbb{R}^n)^+)$ so that Ω contains all of the fixed points of $E(p,\cdot)$ for an arbitrary $p \in [0,1]$. The only fixed point of $E(0,\cdot)$ is $f_i = A_i(0) = B_i(0)$ and $\xi = 0$, and it is clear that the linearisation of $E(0,\cdot)$ at this solution is trivial, so we find that $deg(I - E(0,\cdot), \Omega, 0) = 1$. Property (ii) of Theorem 4.1.1 then implies that $deg(I - H(0,\cdot), \Omega, 0) = deg(I - E(1,\cdot), \Omega, 0) = 1$ as well. □

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Appendix A

Theorems from Mathematical Analysis

In this Appendix, we list some of the standard theorems in mathematical analysis that are referred to in the main text of the thesis.

Theorem A.0.1 (The Arzela-Ascoli Theorem)**.** *Let $f_n : [0,1] \to \mathbb{R}$ be a sequence of continuous functions that are uniformly bounded, i.e., there exists an $M > 0$ such that $|f_n(x)| \leq M$ for each n and $x \in [0,1]$. Suppose in addition that the sequence of functions f_n is uniformly continuous, i.e., for each $\varepsilon > 0$ and $x \in [0,1]$, there exists a δ such that if $|x-y| < \delta$, then $|f_n(x) - f_n(y)| < \varepsilon$. Then there exists a uniformly convergent subsequence $\{f_{n_k}\}_{k=1}^{\infty}$.*

Corollary A.0.2. *Fix a non-negative integer k. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions bounded in $C^{k+1}[0,1]$, then there is a subsequence which is convergent in $C^k[0,1]$.*

Theorem A.0.3 (The Banach Fixed Point Theorem)**.** *Let X be a complete metric space and choose a function $f : X \to X$. Suppose there exists an $M \in (0,1)$ such that $d(f(x), f(y)) \leq Md(x,y)$ for each $x, y \in X$. Then there exists a unique $x \in X$ such that $f(x) = x$.*

Theorem A.0.4 (Sard's Theorem)**.** *Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1. Let $X = \{x \in \mathbb{R}^n : J(f)(x) = 0\}$, where $J(f)(x)$ is the Jacobian determinant of f evaluated at x. Then $\mathbb{R}^n \setminus f(X)$ is dense in $\mathbb{R}^n$.*

Theorem A.0.5 (The Stone-Weierstrass Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded and open, and choose a continuous function $f : \bar{\Omega} \to \mathbb{R}^n$. Then for each $\varepsilon > 0$, there exists a continuously differentiable function $\tilde{f} : \bar{\Omega} \to \mathbb{R}^n$ (in fact, we can choose $\tilde{f}$ to be a polynomial) such that $||\tilde{f} - f||_{C^0(\bar{\Omega})} < \varepsilon$.*