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Thermal dispersion effects on forced convection in a porous-saturated pipe

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ABSTRACT
Thermal dispersion effects on fully developed forced convection inside a porous-saturated pipe are investigated. The pipe wall is assumed to be kept at a uniform and constant heat flux. Having the fully developed velocity field furnished by an arbitrary power series function, the energy equation is solved using asymptotic techniques for the limiting case when thermal conductivity, as a result of thermal dispersion, weakly changes with the Péclet number. A numerical solution, valid for the entire range of thermal dispersion conductivity, is also presented. This latter solution is presented to check the accuracy of the former. The two solutions are then cross-validated in the limits. Besides, results are found to be in good agreement with those previously reported in the literature.

Keywords: Thermal dispersion, asymptotic technique, numerical, cylindrical tube, porous media.

Nomenclature
\begin{align*}
a, b, c & \text{ coefficients of the series [-]} \\
B & \text{ dimensionless constant [-]} \\
C & \text{ perturbation parameter [-]} \\
C_d & \text{ thermal dispersion coefficient [-]} \\
C_F & \text{ form drag coefficient [-]} \\
\epsilon_p & \text{ specific heat at constant pressure [J/kg K]} \\
D_1 & \text{ integration constant [-]} \\
Da & \text{ Darcy number [-]} \\
F & \text{ dimensionless form drag coefficient [-]} \\
G & \text{ applied pressure gradient [Pa/m]} \\
I & \text{ Bessel function [-]} \\
k & \text{ thermal conductivity [W/m.K]} \end{align*}
$K$  permeability [m$^2$]
$M$  $\mu_{\text{eff}} / \mu$ [-]
$\text{Nu}$  Nusselt number [-]
$\text{Pe}$  Péclet number, $\text{Pe} = \text{RePr}$ [-]
$\text{Pr}$  Prandtl number, $\text{Pr} = \frac{\mu_c \rho}{k_f}$ [-]
$R$  pipe radius [m]
$\text{Re}$  Reynolds number, $\text{Re} = \frac{\rho U \sqrt{K}}{\mu}$ [-]
$q''$  wall heat flux [W/m$^2$]
$s$  $(M \text{Da})^{-1/2}$ [-]
$T^*$  temperature [K]
$T_m$  bulk mean temperature [K]
$T_w$  downstream wall temperature [K]
$u$  $\mu u^*/GR^2$ [-]
$u^*$  filtration velocity [m/s]
$\hat{u}$  $u^*/U$ [-]
$U$  mean velocity [m/s]
$(x^*, r^*)$  coordinate system [m]
$r$  $r^*/R$ [-]

Greek symbols

$\theta$  $(T^* - T_w)/(T_m - T_w)$ [-]
$\phi$  porosity [-]
$\kappa$  thermal conductivity ratio $k_f/k_e$
$\mu$  dynamic viscosity [Pa.s]
$\mu_{\text{eff}}$  effective dynamic viscosity [Pa.s]
$\psi$  modified dimensionless temperature [-]
$\rho$  fluid density [kg/m$^3$]

Subscripts

$(0), (1)$  term sequence in asymptotic expansion
1. Introduction

Because of its relevance to a variety of engineering applications including geothermal systems, underground fire control, coal and grain storage, solid matrix heat exchangers, and energy recovery in high temperature furnaces, convection in porous media is a well-developed field of investigation [1-3]. Porous heat exchangers were investigated for their possible applications in solar thermal plants [4], cooling towers [5], electronic cooling [6], exhaust gas recirculation for diesel engines [7] and thermal storage systems [8]. The effects of thermal dispersion on convection in porous media have been analyzed in details as surveyed in [9]. Closed form solutions, to fully developed thermal energy equation, can be obtained for the case of Darcy flow. However, with the boundary, inertia and convective term effects included in the fully developed momentum transfer equation in porous media, no analytical solution has been reported in the literature. Furthermore, thermal analysis of the problem has to rely on a prescribed velocity field. This is because thermal dispersion conductivity is a function of the (volume-averaged) fluid velocity which is not uniformly distributed over the duct cross-section. In fact, a core region is observed away from the walls while the velocity sharply changes near the walls [10]. Hence, analytical solution to the temperature distribution can be obtained as a combination of log, hyperbolic and polylogarithm functions with imaginary arguments [9] which are too complex to be useful in engineering applications where a quick assessment of heat transfer through porous media is of primary interest. Numerical simulations have been exclusively used in the literature for such non-Darcy flow problems [11-14]. Hunt and Tien [15] experimentally studied non-Darcian forced convection flow and heat transfer in high-porosity fibrous media. A model was put forward for thermal dispersion and the adequacy of a homogeneous energy equation to model the transport was ascertained.

Thermal dispersion tensors were calculated within an infinite porous medium formed by a spatially periodic array of longitudinally-displaced elliptic rods by Pedras and de Lemos [16]. The authors applied a low Reynolds $k-e$ closure for turbulence and investigated the effects of solid-fluid thermal conductivity ratio using a unit-cell geometry in conjunction with periodic boundary conditions for mass, momentum and energy equations. Cell-integrated results indicated that compared to the longitudinal dispersion coefficient, the transversal counterpart is more sensitive to porosity, the applied boundary condition type, medium morphology and solid–fluid conductivity.
Two methods of volume average and multiple scale expansion were undertaken to model the thermal dispersion in a rigid homogeneous porous medium described by a periodic model in [17]. The theoretical longitudinal thermal dispersion coefficient for a stratified system was found to be in good agreement with those obtained through the use of a random walk method.

Cheng [18] investigated fully-developed flow in a rectangular and an annular packed bed using Van Driest’s mixing length theory. The predicted heat transfer features were contrasted with the experimental data to obtain the constants in the mixing length theory as well as those in the expression for the transverse thermal dispersion.

Ozgumus and Mobedi [19] numerically investigated the effects of pore to throat size ratio on thermal dispersion of periodic porous media consisting of inline array of rectangular rods. The difference between macroscopic and microscopic values of temperature and velocity are computed numerically so that the thermal dispersion coefficients of the porous media can be determined. It was reported that for Re > 10, higher Reynolds number and porosity values increase both the transverse and longitudinal thermal dispersion coefficients. Interestingly, an optimum value for pore to throat size ratio was reported maximizing the longitudinal thermal dispersion coefficient.

Metzeger et al. [20] have evaluated the thermal dispersion coefficients for water flow through a packed bed of glass spheres by placing thermocouples in the downstream neighborhood of a line heat source to measure the temperature response to a step heat input. Monte Carlo simulations of measurements was performed to quantify the errors. Interestingly, it was reported that the assumption of the one-temperature model is reasonable even in the case of local thermal non-equilibrium.

Ozgumus et al. [21] reviewed the experimental studies conducted to determine the effective thermal conductivity of one class of porous media begin packed beds. The authors categorized the experimental works into three groups: (1) heating/cooling of the lateral boundaries, (2) heat addition at the channel inlet/outlet, (3) internal heat generation. Experimental details, methods, obtained results, and suggested correlations for the determination of the effective thermal conductivity were presented.

In order to by-pass the difficulty in the analysis of this problem, an asymptotic solution is presented here based on suggestions and simplifications discussed in [22]. These make our asymptotic solution valid for a range of thermal dispersion conductivity values. In forthcoming sections, the analysis of the problem is discussed along with numerical solutions for a wide range of thermal dispersion conductivity values. Comparison between the two solutions, sets the range of validity of the theoretical results obtained based on the asymptotic techniques.

2. Analysis

For the steady-state fully developed flow, we have unidirectional flow in the $x^*$-direction inside a porous-saturated pipe with impermeable wall at $r^* = R$, as illustrated in Figure 1.
For $x^* > 0$, the (downstream) heat flux at the tube wall is held constant at the value $q''$. The momentum equation [9] is

$$\mu_\text{eff} \left( \frac{d^2 u^*}{dr_*^2} + \frac{1}{r^*} \frac{du^*}{dr_*} \right) - \frac{\mu u^*}{K} - \frac{C_F \rho u^*}{\sqrt{K}} + G = 0, \quad (1)$$

where $\mu_\text{eff}$ is an effective viscosity, $\mu$ is the fluid viscosity, $K$ is the permeability, and $G$ is the applied pressure gradient. The dimensionless variables are defined as

$$x = \frac{x^*}{\text{Pe} R}, \quad r = \frac{r^*}{R}, \quad u = \frac{\mu u^*}{GR^2}. \quad (2)$$

The dimensionless form of Eq. (1) is then given by

$$\frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} - s^* u - F s^* \frac{1}{M} = 0. \quad (3)$$

The viscosity ratio $M$, the Darcy number $Da$, shape factor $s$ and form drag coefficient $F$ are defined by

$$M = \frac{\mu_\text{eff}}{\mu}, \quad Da = \frac{K}{R^2}, \quad s = \frac{1}{\sqrt{MDa}}, \quad F = \frac{C_F \rho GR^3}{\mu \mu_\text{eff}}. \quad (4)$$

Equation (3) can be solved subject to no slip boundary condition at the wall and the symmetry condition at the pipe center. One can assume that, once solved, the solution can be expressed in terms of a finite number of terms as follows.
\[ u = \sum_{n=0} a_n r^n. \]  

(5)

In fact, recast of a continuous function using Taylor series expansion in the form of Eq. (5) is well documented in the literature and does not need to be reiterated here.

The mean velocity \( U \) and the bulk mean temperature \( T_m \) are defined by

\[ U = \frac{2}{R^2} \int_0^R u^* r^* dr^*, \quad T_m = \frac{2}{R^2 U} \int_0^R u^* T^* r^* dr^*. \]  

(6-a, b)

Further dimensionless variables are defined by

\[ \hat{u} = \frac{u^*}{U} = \frac{\sum_{n=0} a_n r^n}{\sum_{n=0} a_n (n+2)}, \]  

(7)

and

\[ \theta = \frac{T^* - T_w}{T_m - T_w}. \]  

(8)

The Nusselt number \( Nu \) is defined as

\[ Nu = \frac{2 R q^*}{\kappa_e (T_w - T_m)}. \]  

(9)

Homogeneity and local thermal equilibrium is assumed see Nield [23] for criteria under which this latter assumption is valid. The steady state thermal energy equation in the absence of viscous dissipation, heat source terms and axial conduction is

\[ \rho c_p u^* \frac{\partial T^*}{\partial \xi^*} = \frac{1}{r^*} \frac{\partial}{\partial r^*} \left( kr^* \frac{\partial T^*}{\partial r^*} \right), \]  

(10)

where the thermal conductivity is given as the sum of stagnant (effective) and dispersion conductivity as

\[ k = k_s + k_d. \]  

(11)

These can be related to fluid, flow and porous matrix properties as

\[ k_s = k_{f, eff} + k_{s, off} \]  

\[ k_d = \kappa_s C_s \text{ Re Pr } \hat{\kappa} \hat{u} \]  

(12a,b)

Note that a number of correlations are available to evaluate the fluid and the solid phase thermal conductivity and thermal dispersion conductivity [9]. Nonetheless, in our parametric study in this paper, we use the above equations which are generic enough to be valid for any of the existing models in the literature. In either case, regardless of the applied thermal conductivity model, the first law of thermodynamics leads to
\[ \frac{\partial T^*}{\partial r^*} = \frac{q^*}{\rho c_p 2RU}. \]  

(13)

Applying the above equation and recast in non-dimensional form, after combining equations (11) and (12) with Eqn. (10), the energy equation becomes

\[ \frac{d}{dr} \left( r(1 + C\hat{u}) \frac{d\theta}{dr} \right) = -r\hat{u}Nu, \]  

(14)

where the boundary conditions are

\[ \frac{d\theta}{dr} \bigg|_{r=0} = 0 \quad \text{and} \quad \theta \bigg|_{r=1} = 0. \]  

(15a,b)

and the asymptotic parameter \( C \) is defined as

\[ C = C_d \Re \Pr \kappa \]  

(16)

In their experimental analysis of air flow through aluminum foam heat exchangers, Calmidi and Mhajan [12] reported a value of 0.06 for their \( C_d \) while their \( \Re \) (here \( \Re \) is pore-Reynolds, i.e. square root of permeability is defined as the length scale) was limited to 135. The maximum numerical value for the group \( \Pr \kappa \) in that paper was reported to be 0.0075. This combination leads to a maximum of 0.061 for \( C \). Hunt and Tien [15], on the other hand, reported theoretical and experimental analysis of dispersion effects in foams with water as the working fluid leading to 0.6 as their maximum \( C \) value. Therefore, we also limit the numerical value of \( C \) to 0.6 in this work.

3. Asymptotic Solutions

Eq. (14) can be integrated once, using direct integration, to result in

\[ \frac{d\theta}{dr} = -Nu \frac{1}{1 + C\hat{u}} \sum_{n=0} \frac{b_n r^{n+1}}{n+2} + \frac{D_i}{r}, \]  

(17)

The symmetry boundary condition for the temperature, Eq. (15a), results in \( D_i = 0 \) in Eq. (17) above. As such, one can rewrite the above equation as

\[ \frac{d\theta}{dr} = \left(-Nu \sum_{n=0} \frac{b_n r^{n+1}}{n+2} \right). \]  

(18)

Considering the case of small \( C \) values \((C \ll 1)\), as was the case in the experiments reported in [12], one can simplify Eq. (18), by expanding the denominator, to get

\[ \frac{d\theta}{dr} \approx -Nu \sum_{n=0} b_n r^{n+1} \left(1 - C \sum_{n=0} b_n r^n\right). \]  

(19)

The above equation can be solved using the following asymptotic expansion for the temperature distribution
\[ \theta = \theta_0 + C \theta_1 + \ldots \]  
(20)

A two-term expansion has been used here similar to [22] and terms smaller than \(O(C^2)\) are neglected. Here, \(C\) is chosen as our perturbation parameter. Obviously, the results are only valid for \(C << 1\) which for most cases of practical interest, for which experiments are conducted, this requirement is met. One can now obtain the zeroth and first order solutions as

\[
\frac{d\theta_0}{dr} \approx -Nu \sum_{n=0} b_n r^{n+1} \\
\frac{d\theta_1}{dy} \approx rNu \sum_{n=0} b_n r^n \sum_{n=0} b_n r^n = Nu \sum_{n=0} c_n r^{n+1} 
\]

(21a,b)

The zeroth order solution reads

\[
\theta_0 = -Nu \sum_{n=0} b_n \left( r^{n+2} - 1 \right),
\]

(22)

The first order solution can be obtained noting that

\[
\frac{d\theta_1}{dy} \approx rNu \sum_{n=0} c_n r^n,
\]

\[
c_n = \sum_{k=0}^{n} \frac{b_k b_{n-k}}{k + 2}
\]

(23a,b)

according to Cauchy product of two power series. Note that \(b_n\) values are known as they are obtained by expanding the velocity distribution function according to Eq. (7). With \(F=0\), the closed form solution for the velocity profile is given in [9] where it is shown that modified Bessel functions \((I_j)\) furnish the velocity field, i.e.

\[
\hat{u} = \frac{I_0(s) - I_0(sr)}{I_0(s) - 2I_1(s)/s}.
\]

Expanding the combination of zeroth and first order Bessel functions is a straightforward task as described in [22]. With non-zero \(F\) values, approximate solutions are offered in the literature, see [24] for instance, which are already expressed in terms of polynomials, i.e. for \(s<<1\) one has

\[
\hat{u} = 2(1 - r^2) + \frac{Fs}{144M} (2r^6 - 9r^4 + 9r^2 - 2)
\]

and for \(s>>1\)

\[
\hat{u} = (1 + \frac{2}{s})(1 - e^{-s(1-r)})
\]

In either case, for a given \(F\) value, with known \(b_n\) values one can simply use Eq. (23b) to obtain the \(c_n\) values. This makes integration of Eq. (23a) possible leading to

\[
\theta_1 = Nu \sum_{n=0} \frac{c_n}{n+2} \left( r^{n+2} - 1 \right)
\]

(24)
Combining Eq. (7), (22) and (24), one has the temperature profile as

\[ \theta = -N u \sum_{n=0} b_n \left( \frac{b_n}{(n+2)^2} - C \frac{c_n}{n+2} \right) \left( r^{n+2} - 1 \right), \]  

(25)

Finally, the Nusselt number \( Nu \) can be found by substituting for \( \hat{u} \) and \( \theta \) - using Eqs. (7) and (25) - in the compatibility condition

\[ 2 \int_0^1 r \hat{u} \theta \, dr = 1, \]  

(26)
as a function of \( s, M, F \) and \( C \), which takes the following form

\[ Nu = \frac{-1}{2 \int_0^1 \left( \sum_{n=0} b_n \left( \frac{b_n}{(n+2)^2} - C \frac{c_n}{n+2} \right) \left( r^{n+2} - 1 \right) \right) \left( \sum_{n=0} b_n r^n \right) t d r}, \]  

(27)

### 4. Numerical Solution Procedure

A finite difference central scheme was applied to solve the momentum and thermal energy equations; both ordinary differential equations. The dimensionless momentum equation was linearized similar to [25] by replacing the nonlinear term in each iteration by the product of the velocity in that same and previous iteration, i.e. \( u_{\text{new}}^2 = u_{\text{old}} u_{\text{new}} \) where the subscript “new” denotes the velocity in each iteration while “old” refers to the velocity at the same spatial location obtained in the previous iteration. With a converged solution, the difference between velocity profile from two successive iterations is negligible hence the original equation is consistently recovered, i.e. the difference between the “new” and “old” velocity values is less than the convergence criterion (here set at \( 10^{-4} \)) hence \( u_{\text{new}}^2 = u_{\text{new}} u_{\text{new}} \).

Having found the solution for the resulting system of linear algebraic equations -by subroutine “SY” as mentioned in [1] - the mean velocity will be found by means of Eq. (6-a). This integral is solved by midpoint rule -as explained by [26] - to give \( \hat{u} \) which is required for solving Eq. (14). We solve for a modified dimensionless temperature defined as \( \psi = \frac{\theta}{Nu} \) to uncouple Eqns. (14) and (26) similar to [1] where the dimensionless temperature was normalized by the Nusselt number. The boundary conditions remain the same as those in Eq. (15a,b).

The resulting equation is then solved in a similar manner to that we applied to solve the momentum equation. The compatibility condition is then solved by midpoint rule to give the Nusselt number. The numerical solution is readily complete and we can compare the results of the two methods. A total of 30 grid points were used as it was noted that moving to 60 and 90 points would not significantly change the results. In fact, the velocity and temperature profiles were found to be indistinguishable when moving from 60 to 90 grid points. The change in the maximum velocity and
temperature values for a number of test runs we conducted was less than 5%. Furthermore, numerical results are validated against the limiting cases, of $s=0$, $F = 0$, and $s→∞$ with $C = 0$ for which exact solution can be found as listed in Nield and Bejan [9], to observe an excellent agreement (within two decimal points for Nu values). With that, the numerical solution is readily complete and we can compare the results of the two methods for given $C$, $F$ and $s$ values.

5. Results and Discussion

One can verify theoretical results by comparing them with the data available in the literature. For instance when $F = 0$, and $s→∞$ with $C = 0$ for which the velocity tends to Darcy flow, dimensionless temperature can be expressed as a parabolic function of $r$ leading to $Nu = 8$; see [9]. For all cases considered here the Brinkman term is maintained in the momentum equation unless mentioned otherwise. To show the accuracy of the procedure offered here, we rearrange the velocity and temperature as truncated power series, i.e. $b_0 = 1$ and for $n > 1$ we have $b_n = 0$ in Eq. (7). Using Eq. (22), our temperature field is described as

\[
\theta = \frac{Nu}{4} (r^2 - 1),
\]

\[
\theta = \frac{Nu}{2} (r^2 - 1),
\]

\[
\theta = \frac{Nu}{4} (1 - r^2) - C \frac{Nu}{2} (1 - r^2).
\]

The zeroth order term, $\theta_0$, is exactly the same as the one reported in [1] while the first order solution makes it different from that of [1]. The compatibility condition, Eq. (27), now reads

\[
Nu = \frac{1}{2} \int \left( \frac{1}{4} (1 - r^2) - C \frac{1}{2} (1 - r^2) \right) \, dr,
\]

which leads to

\[
Nu = 8(1 + 2C),
\]

with $C = 0$, the solution tends to $Nu = 8$ as expected. With non-zero $C$ values, interestingly, the solution shows a linear increase of $Nu$ with $C$. This trend is in-line with the observations made in [28-30]. In fact, recasting Eq. (29b) as follows

\[
\frac{Nu}{Nu_{no\text{-}dispersion}} = 1 + 2C,
\]

the Nusselt number ratio becomes independent of the duct cross-sectional shape when the right-side of Eq. (29c) is compared with Eq. (27) of [28]. The results will remain the same if non-zero $F$ values are
tested, i.e. when the form drag effects are to be included in the absence of the Brinkman term. This is because the velocity profile will remain as a plug flow.

With the inclusion of Brinkman term, however, things will change and there will be a near-wall region with a velocity-gradient which will vary depending on the $M$, $s$ and $F$ values. For instance, an interesting solution can be recovered with non-zero values of $F$ for which [24] offers an asymptotic solution for both small and large $s$ values. Taking small $s$ values as an example, one has

$$\hat{u} = 2 - 2B + (9B - 2)r^2 - 9Br^4 + 2Br^6$$

(30)

with

$$B = \frac{Fs}{144M}$$

(31)

With $C = 0$, Eq. (22) will reproduce the dimensionless temperature (and consequently the Nusselt number) therein. With non-zero values of $C$, however, one has to use Eq. (23) to find the $c_n$ values for each combination of $F$ and $s$ (note that we assumed $M = 1$ throughout this work). One of the reviewers emphasized on the fact that a model for estimating the effective viscosity is not agreed on in porous media literature; see [31-35]. To present a unifying model/theory to accurately predict $M$, is beyond the scope of this paper and, in this work, we follow a number of studies assuming $M = 1$; see [9] for a comprehensive list. Table 1 shows the coefficients that should be used to prescribe the velocity and temperature field using Eqns. (7) and (22). Note that $c_n$ values are from Eq. (23). Furthermore, one is reminded that with $n > 12$, one has $c_n = 0$ because the contributing $b_n$ values are already equal to zero.

<table>
<thead>
<tr>
<th>n</th>
<th>$b_n$</th>
<th>$c_n$</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>$s = 1$, $F = 0.1$, $C = 0.1$</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1.998611</td>
<td>0.399792</td>
</tr>
<tr>
<td>2</td>
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<tr>
<td>4</td>
<td>-0.00625</td>
<td>-0.0166</td>
</tr>
<tr>
<td>6</td>
<td>0.001389</td>
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</tr>
<tr>
<td>8</td>
<td>0</td>
<td>1.29E-05</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
<td>2.11E-08</td>
</tr>
<tr>
<td>12</td>
<td>0</td>
<td>-1.7E-09</td>
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<tr>
<td></td>
<td>$s = 1$, $F = 1$, $C = 0.1$</td>
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The above are used to generate temperature profiles and Nusselt numbers for non-zero $C$ values. Figure 2 shows the dimensionless temperature profile as a function of $r$ for different $C$ values. Note that our results were indistinguishable from those in the literature for $C = 0$ and therefore no comparisons were presented for that limiting case in this figure. Adding to numerical values of $C$, the dimensionless temperature profiles are flattened indicating higher heat transfer rates at the wall. The near-wall temperature profile shows a much sharper gradient while away from the wall the maximum dimensionless temperature is dropped compared to the case when thermal dispersion is negligible ($C = 0$). In the extreme case, when $C = 0.6$, the maximum dimensionless temperature drops by just under 50% compared to no-dispersion case, i.e. when $C = 0$. 

\[
\begin{array}{|c|c|c|}
\hline
8 & 0 & 0.000118 \\
10 & 0 & 2.11E-06 \\
12 & 0 & -1.7E-07 \\
\hline
\end{array}
\]
s = 0.5, $F = 1$, $C = 0.1$

\[
\begin{array}{|c|c|c|}
\hline
0 & 1.993056 & 0.398957 \\
2 & -1.96875 & -0.04948 \\
4 & -0.03125 & -0.01633 \\
6 & 0.006944 & 2.19E-05 \\
8 & 0 & 6.21E-05 \\
10 & 0 & 5.27E-07 \\
12 & 0 & -4.3E-08 \\
\hline
\end{array}
\]
s = 0.5, $F = 1$, $C = 0.5$

\[
\begin{array}{|c|c|c|}
\hline
0 & 1.993056 & 0.00173 \\
2 & -1.96875 & 0.244812 \\
4 & -0.03125 & -0.07816 \\
6 & 0.006944 & -0.00684 \\
8 & 0 & 0.00031 \\
10 & 0 & 2.64E-06 \\
12 & 0 & -2.2E-07 \\
\hline
\end{array}
\]
Eq. (28c) presents a closed form solution for the simple case of plug flow with thermal dispersion effects included in the thermal energy equation. Results of applying this approximate equation are presented in Fig. 3 versus numerical simulation of the problem for different $s$ values with $F = 0$ and $C = 0.1$. Note that Eq. (28c) is valid for the limiting case when $s \to \infty$. As seen, there is a slight difference between the maximum value for the dimensionless temperature profile at the pipe centreline. This difference is less pronounced for higher $s$ values. For $s = 10$, the difference between the numerical and theoretical solution is under 5% which is excellent given the simplicity in using Eq. (28c).
Finally, one is interested in evaluating the effects of thermal dispersion on the overall heat transfer represented by the Nusselt number. Figure 4 shows our Nusselt number versus $C$ for different extreme $s$ values. For very high $s$ values, as Eq. (29b) predicts, $Nu$ linearly varies with $C$ while for lower $s$ values we need to use either the numerical or theoretical procedure presented here. We chose the latter to generate the data for which Fig. 4 is plotted. Note that we do not expect the results for different $s$ values to be close mainly because the velocity fields are different. For $s=0.5$, the velocity profile is closer to that of clear flow and resembles a parabola while with $s=100$ the velocity profile is flattened and looks like that of the Darcy flow except for a near wall region where velocity gradient is sharp. Hence, Fig. 4 only shows different trends in $Nu$ versus $C$ for different $s$ values.
Figure 4 The Nusslet number versus C for different s values with F = 2.

Note that in both cases, thermal dispersion leads to higher heat transfer as manifested in higher \( Nu \) values with an increase in \( C \). While mathematically it is shown that or high \( s \) values a linear increase in \( Nu \) is expected for lower \( s \) values, the trend here does not look like a linear one despite the fact that a linear extrapolation is extremely desirable for its simplicity. To assess that possibility, we present Fig. 5 that shows the Nusselt number divided by that of no-dispersion case on the ordinate while \( C \) is plotted on the abscissa. Numerical values of the parameters are the same as those in fig. 4. Interestingly, the use of a linear function leads to only 15% error for the range of \( C \) values considered here. One however, is warned that this may or may not hold if much higher \( C \) values are used noting that our asymptotic expansion is only valid for \( C << 1 \). Having said that, it is still a notable observation that a linear function, Eq. (29c), can be applied to predict the Nusselt number ratio which can then be used to obtain the actual \( Nu \) value based on that of no-dispersion which is available in the literature; see [36]
6. Conclusion

Thermal dispersion effects on fully developed forced convection in a porous-saturated pipe are investigated using numerical and asymptotic techniques. The latter relies on expanding the fully developed velocity profile in terms of a power series allowing for a series solution for the local temperature profile. With known velocity and temperature field, one can then obtain the Nusselt number as a function of the key parameters including the thermal dispersion coefficient. The results from the two approaches are cross-validated. It was noted that using a plug flow assumption and obtaining a simple closed form solution for the temperature distribution can be accurate with a maximum error of about 5%. The Nusselt number is found to be a linear function of thermal dispersion coefficient for plug flow. It is noted that applying a similar correlation to a problem where the Brinkman term is retained can lead to 15% error in predicting the Nusselt number ratio using our Eq. (29-c).

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