

## Photon-added detection

A. M. Brańczyk,<sup>1</sup> Tobias J. Osborne,<sup>1,2</sup> Alexei Gilchrist,<sup>1,\*</sup> and T. C. Ralph<sup>1</sup>

<sup>1</sup>Centre for Quantum Computer Technology and Department of Physics, University of Queensland, Queensland 4072, Australia

<sup>2</sup>Department of Mathematics, University of Bristol, United Kingdom

(Received 30 May 2003; revised manuscript received 18 August 2003; published 20 October 2003)

The production of conditional quantum states and quantum operations based on the result of measurement is now seen as a key tool in quantum information and metrology. We propose a different type of photon number detector. It functions nondeterministically, but when successful, it has high fidelity. The detector, which makes use of an  $n$ -photon auxiliary Fock state and high efficiency homodyne detection, allows a tunable trade-off between fidelity and probability. By sacrificing probability of operation, an excellent approximation to a photon-number detector is achieved.

DOI: 10.1103/PhysRevA.68.043821

PACS number(s): 42.50.Dv

### I. INTRODUCTION

In quantum theory, measurements encapsulate our observation of nature. They are the link between the abstract machinery of the theory and its observational consequences. Because of this, it is not surprising that often new measurement techniques and strategies can drive new applications. Moreover, the production of conditional quantum states and quantum operations based on the results of measurement is now seen as a key tool in realizing quantum information processing goals [1,2]. In optical schemes, conditional measurements provide an effective nonlinearity that allows optical quantum gates to be fashioned [1,3–6], and the creation of highly entangled states suitable for quantum metrology [7–10].

Often, however, the ideal measurements envisioned in theoretical proposals are not so easily realized experimentally. Linear optics quantum computation schemes such as in Ref. [1], require high efficiency *selective* detectors (detectors able to distinguish between zero, one, and several photons). The most promising detector candidate in this regard is the visible-light photon counter (VLPC) [11,12] which has achieved efficiencies of the order of 88%. Unfortunately these detectors require extreme operating conditions and suffer from high dark-count rates.

In this paper we introduce the idea of a nondeterministic detector based on *photon-added detection* (PAD), where we make use of high efficiency homodyne detection and mix the input state with an  $|n\rangle$  Fock state prior to detection. This detector works non-deterministically, and there is an essential trade-off between the probability that the detector works and the degree to which the detector functions as an  $n$ -Fock state projector. When the detector fails, this is clearly signaled in the output. The essence of the detecting scheme is based on the observation that if we use homodyne detection and postselect within a narrow band of  $2\Delta$  around  $x=0$ , then the detection will only be sensitive to even photon numbers, see Fig. 1. By careful use of quantum interference, we can make the detector act like a projector onto a particular photon number.

The structure of the paper is as follows. First we will introduce the scheme in general, then focus on the limiting case where  $\Delta=0$  to motivate its function. We then consider the effect of a finite  $\Delta$  and discuss the trade-off between probability of operation and fidelity. Finally, before concluding, we examine the effect of detector inefficiencies in our scheme.

### II. THE SCHEME

In order to characterize how well the detector functions we shall calculate the ability of the detector to pick out an appropriate state  $|a_p\rangle$  from an entangled state of the form

$$|\psi\rangle = \mathcal{N}_0 \sum_{n=p-w}^{p+w} |a_n\rangle_a |n\rangle_b \quad (1)$$

when we measure mode  $b$ . The normalization is  $\mathcal{N}_0 = 1/\sqrt{2w+1}$ , and the parameter  $w$  defines a window of states, from which we want to pick out the central component. The reason for choosing this comparison is twofold. First we are interested in states precisely of the above form where the states  $|a_n\rangle$  represent multimode states which we are conditioning by detection and postselection. Second, this approach provides an easily computable measure of how

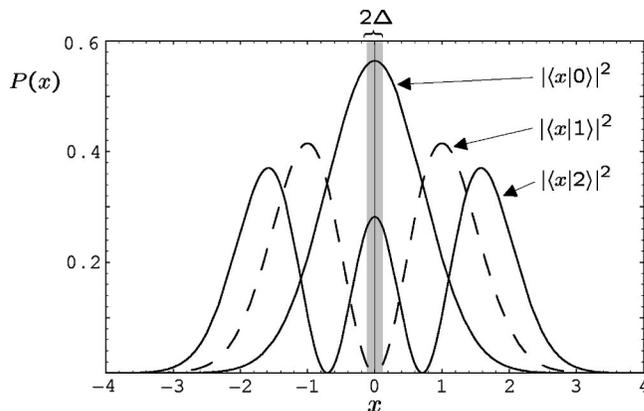


FIG. 1. The probability density of getting a particular  $x$  value if we measure the  $X$  quadrature using homodyne detection. Results are shown for various initial Fock states.

\*Electronic address: alexei@physics.uq.edu.au

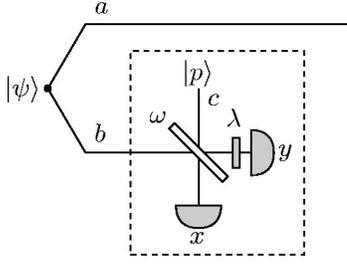


FIG. 2. Quantum circuit describing our detector arrangement.

close to a  $|p\rangle\langle p|$  projector the detector functions in this context, since this approach reduces to a characterization of state preparation [13].

With this characterization in mind, consider the circuit in Fig. 2. We have some multi-mode state  $|\psi\rangle$ , and we wish to condition the state of mode(s)  $a$  dependent on a photon-number measurement on mode  $b$ . For simplicity consider only a single  $n$ -photon Fock state component in mode  $b$ , the general case is recovered through additivity, i.e.,  $|\psi\rangle = \mathcal{N}_0 \sum_n |\psi^{(n)}\rangle$ . The input state is then some state  $|\psi^{(n)}\rangle = |a_n\rangle_a |n\rangle_b |p\rangle_c$ , where  $|a_n\rangle$  is the associated component in mode  $a$  and mode  $c$  is initially in a  $p$ -photon Fock state. After interacting on a beam splitter of reflectivity  $\cos^2(\omega)$  and undergoing a phase shift  $\lambda$  on mode  $b$ , the output state is

$$|\psi_{\text{out}}^{(n)}\rangle = \frac{|a_n\rangle_a}{\sqrt{n!p!}} \sum_{m=0}^n \sum_{q=0}^p \binom{n}{m} \binom{p}{q} e^{i\pi(p-q)+i(m+q)\lambda} \times c^{m+p-q} s^{n-m+q} \hat{b}^{\dagger m+q} \hat{c}^{\dagger n+p-(m+q)} |00\rangle_{bc}, \quad (2)$$

where  $\hat{b}^{\dagger}$  and  $\hat{c}^{\dagger}$  are the bosonic creation operators for modes  $b$  and  $c$ , respectively,  $c = \cos(\omega)$ , and  $s = \sin(\omega)$ , and finally we also have the usual binomial coefficients  $\binom{u}{v} = u!/(u-v)!v!$ .

Modes  $b$  and  $c$  are now detected using separate balanced homodyne detectors. To an excellent approximation such detectors can be modeled as projectors onto small ranges of quadrature amplitude eigenstates  $|x_\theta\rangle$  where  $x_\theta$  is a continuous variable with infinite dimension, and  $\theta$  describes the phase relationship with the local oscillator of the homodyne detector. The final conditional state (unnormalized), given we obtain  $x_\theta$  in one detector and  $y_\phi$  in the other is  $|\psi_{\text{cond}}\rangle = \mathcal{N}_0 \sum_n |x, y\rangle \langle x, y | |\psi_{\text{out}}^{(n)}\rangle = \mathcal{N}_0 \sum_n |\psi_{\text{cond}}^{(n)}\rangle$  where

$$|\psi_{\text{cond}}^{(n)}\rangle = \frac{e^{-i(n+p)\phi - 1/2(x_\theta^2 + y_\phi^2)} |a_n, x_\theta, y_\phi\rangle}{\sqrt{n!p! \pi 2^{n+p}}} \times \sum_{m=0}^n \sum_{q=0}^p \binom{n}{m} \binom{p}{q} e^{i\pi(p-q)+i(m+q)(\lambda - \theta + \phi)} \times c^{m+p-q} s^{n-m+q} H_{m+q}(x_\theta) H_{n+p-(m+q)}(y_\phi), \quad (3)$$

here we have used the fact that the overlap between the quadrature amplitude eigenstates and the number states is given by

$$\langle x_\theta | n \rangle = \frac{H_n(x_\theta)}{\sqrt{\sqrt{\pi} 2^n n!}} e^{-(1/2)x_\theta^2 - in\theta} \quad (4)$$

and  $H_n(x)$  is the Hermite polynomial of order  $n$ . We have chosen the convention that  $\theta=0$  quadrature operator can be written in terms of the mode operators as  $X = (a + a^\dagger)/\sqrt{2}$ . Notice that the quadrature phase angles  $\theta$  and  $\phi$  are effectively not independent of  $\lambda$  and that without loss of generality we can absorb those terms into  $\lambda$  (so we will take  $\lambda - \theta + \phi \rightarrow \lambda$ ). For simplicity we shall also take  $\phi=0$  and set the overall phase of this component to zero, and hence we can also drop the quadrature angle subscript on  $x$  and  $y$ . Now consider the case where we use a 50:50 beam splitter so that  $\omega = \pi/4$  and we set  $\lambda = \pi/2$ . With these conditions Eq. (3) reduces to

$$|\psi_{\text{cond}}^{(n)}\rangle = \frac{e^{-1/2(x^2 + y^2)} e^{i\pi p}}{\sqrt{n!p! \pi 2^{n+p}}} g(n, p) |a_n, x, y\rangle \quad (5)$$

$$g(n, p) = \sum_{m=0}^n \sum_{q=0}^p \binom{n}{m} \binom{p}{q} e^{i\pi/2(m-q)} \times H_{m+q}(x) H_{n+p-(m+q)}(y). \quad (6)$$

To see how this detecting scheme is only sensitive to the  $p$ -Fock component we focus on the limiting case of  $\Delta=0$  next.

### III. LIMITING CASE

Consider only the special case where we happen to detect  $x=y=0$  in the homodyne detectors. For these values, we can use

$$H_n(0) = \begin{cases} 0, & n \text{ odd} \\ (-1)^{n/2} n!, & n \text{ even.} \end{cases} \quad (7)$$

This relation implies that only terms with even  $m+q$  will be nonzero, which in turn implies that  $n+p$  must also be even. If we now write  $g \rightarrow [g(n, p) + g'(n, p)]/2$ , where  $g'(n, p)$  simply has the order of the summations reversed, we get

$$g(n, p) = \frac{1}{2} \sum_{m=0}^n \sum_{q=0}^p \binom{n}{m} \binom{p}{q} H_{m+q}(0) H_{n+p-(m+q)}(0) \times e^{i\pi/2(m-q)} (1 + e^{i\pi k}), \quad (8)$$

where we have set  $n=p+2k$  and used the fact that  $m+q$  must be even. From this expression it is clear that terms with odd  $k$  will also vanish. Terms with even  $k>0$  will also vanish—this can be readily verified numerically. This then

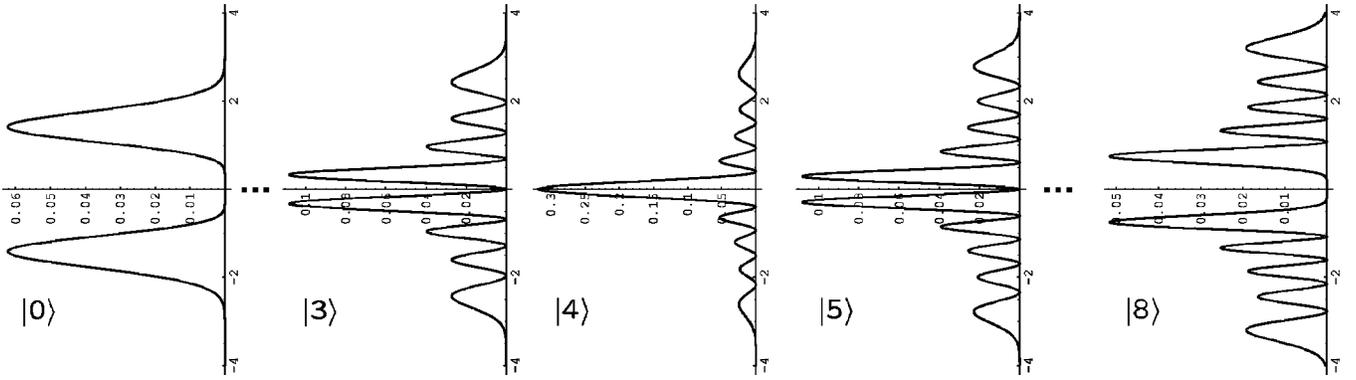


FIG. 3. Probability density plots for obtaining a particular  $x$  and  $y$  for the homodyne detections given an auxiliary photon number of  $|p\rangle=|4\rangle$ , for various input number states  $|n\rangle$ . Only the  $x$  axis shown as the distributions are rotationally symmetric. By postselecting on a narrow band near  $x=0$  the detector becomes only sensitive to components with  $|n\rangle=|p\rangle$ . Also, the noise form having a finite postselection band comes from the nearby number states from the target state.

only leaves the terms with  $k=0$  ( $n=p$ ) as contributing to the state (5) and so the detector picks out the  $|a_p\rangle$  component.

This analysis assumes an infinitesimal acceptance band for the detector. In order to assess the practicalities of the system we need to integrate over some range of values around  $x=y=0$  and evaluate success and failure probabilities. Clearly there will be a trade-off between how well we project onto the  $p$ -photon Fock state and the probability of obtaining a successful outcome.

#### IV. FINITE $\Delta$

The probability density for obtaining a value  $x$  in mode  $c$  and  $y$  in mode  $b$  will be

$$P(x,y) = \text{tr}\{|x\rangle\langle x| \otimes |y\rangle\langle y| \rho\} \quad (9)$$

$$= \text{tr}_a\{\langle x,y|\rho|x,y\rangle\}, \quad (10)$$

where  $\rho$  is the three mode density matrix describing the state after the beamsplitter. This distribution is radially symmetric about the origin, so we will switch to the polar coordinates  $r$  and  $\theta$  (where  $r^2=x^2+y^2$ ) and accept a particular result if it lies within a certain radius  $\Delta$ . Intuitively we can see what the effect will be from Fig. 3. As we make  $\Delta$  larger, the probability that a result falls within the accepted band, picks up contributions from nearby states to the target state, and these will contribute to the error. The total probability that we get  $0 \leq r \leq \Delta$  is

$$P_\Delta = 2\pi \int_0^\Delta P(r,\theta) r dr \quad (11)$$

The (unnormalized) state immediately after destructively obtaining a particular  $x$  and  $y$  in the first two modes is  $\rho_a^{(x,y)} = \langle x,y|\rho|x,y\rangle$ . Consequently the ensemble of states that we would obtain if we were only to accept values within a radius  $\Delta$ , would be

$$\rho_a = \frac{1}{P_\Delta} \int_0^{2\pi} d\theta \int_0^\Delta dr \rho_a^{(r,\theta)} \quad (12)$$

To compare how well such a projector functions we can use the fidelity against the target state  $|a_p\rangle$ ,

$$F(\Delta) = |\langle a_p|\rho_a|a_p\rangle|. \quad (13)$$

Note that in calculating this quantity we will assume that the  $|a_j\rangle$  are orthonormal.

One of the important features of the PAD scheme is that it is sensitive only to a band of number states near the target state. This effect can be seen in the behavior of the probability densities for states far away from the target state in Fig. 3, and is clearly demonstrated in Fig. 4, where we show the rapid convergence in fidelity as we increase the number of nearby states to the one we are projecting out.

As a concrete example, consider projecting out the state  $|a_1\rangle$  from the initial state  $\sum_{j=0}^2 |a_j\rangle|j\rangle$ . The raw probability and fidelity of the resulting state for a given  $\Delta$  are given in Fig. 5.

As we increase  $\Delta$ , the probability that we get a result we will accept also increases, but due to the overlap with the states near the target state the fidelity of the detector will

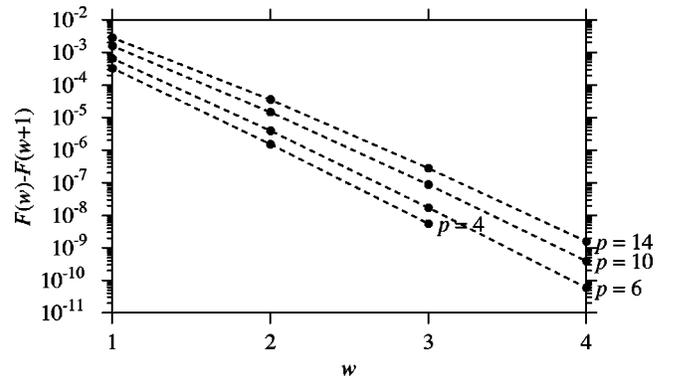


FIG. 4. The main source of error for the detector is due to contributions from number states near the target state. Here we plot the difference in fidelity  $F$  between two successive window sizes,  $w$  and  $w+1$ . As can be seen, increasing the size of the window of states we are testing against makes little difference past a few states, consequently we will adopt  $w=2$  in calculations. Note that  $\Delta=0.1$  in the plots.

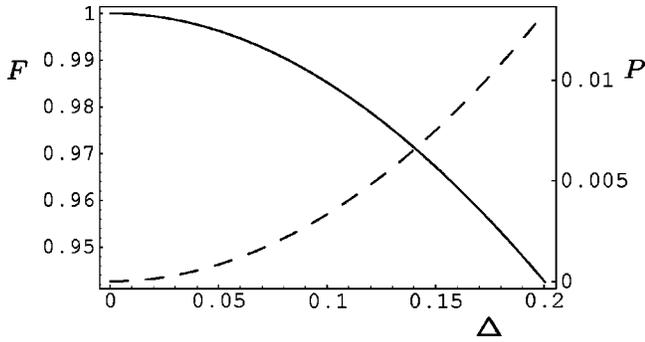


FIG. 5. The probability (dashed line) of obtaining a result within  $\Delta$ , and the fidelity (solid line) with  $|a_1\rangle$  of the resulting state, for an input state  $\sum_{j=0}^2 |a_j\rangle|j\rangle$  and with a single-photon auxiliary.

drop. The actual probability is not a meaningful quantity in this context as it depends as much on the test state (1) as on the parameters of the detector. The quantity we will use instead is a probability rate  $R = P_\Delta / P_{\text{ideal}}$ , which is the probability we get divided by the expected probability if we had an ideal photo counter. The trade-off between fidelity and probability is quantified in Fig. 6.

### V. INEFFICIENT DETECTION

The calculations so far have assumed unit efficiency detection. In this section we explore the effect of nonunit detection efficiencies for the PAD, although it should be noted from the outset that detection efficiency for homodyne detection is very high (in the region of 98% [14]). We will compare the performance of the PAD to an ideal, but inefficient photon counter, which we model by the positive-operator-valued-measure (POVM) elements  $\Pi_p$ :  $p=0,1,\dots$ , where  $p$  is the number of detected photons, with

$$\Pi_p = \sum_{m=p}^{\infty} \binom{m}{p} \eta^p (1-\eta)^{m-p} |m\rangle\langle m|. \quad (14)$$

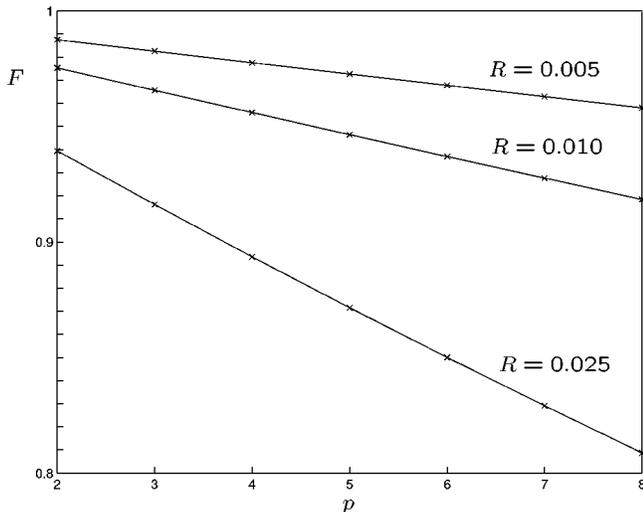


FIG. 6. The fidelity of operation for various target states  $|a_p\rangle$  from the distributions (1) is given (with  $w=2$ ). The curves are for fixed probability rates  $R$ .

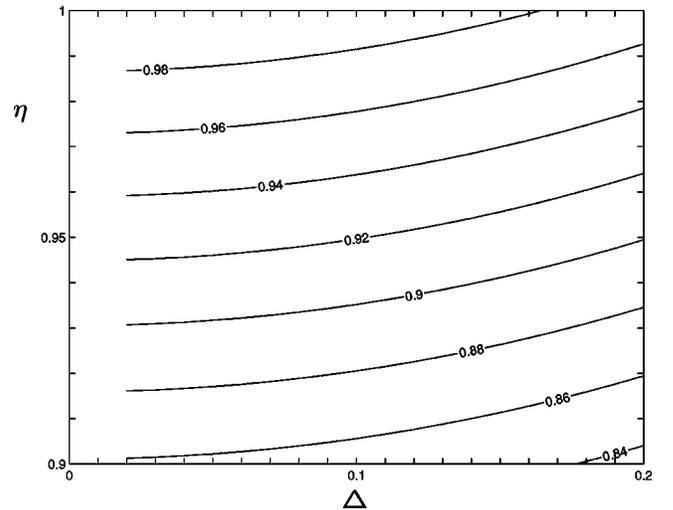


FIG. 7. Equivalent ideal single-photon detector efficiency as a function of the acceptance width  $\Delta$ , and the homodyne efficiency  $\eta$ .

Visible-light photon counters can be modeled as ideal, but inefficient photon counters, at least for small photon numbers [15].

The fidelity of the ideal detector in picking out the state  $|a_p\rangle$  when used with the input state (1) is then

$$F_{\text{ideal}} = \frac{|\langle a_p | \text{Tr}_b \{ \Pi_p \rho_{\text{in}} \} | a_p \rangle|}{\text{Tr} \{ \Pi_p \rho_{\text{in}} \}} = \left( \sum_{n=p}^{n_{\text{max}}} \binom{n}{p} (1-\eta)^{n-p} \right)^{-1}, \quad (15)$$

where the summation extends to the maximum photon number, so for the test state in Eq. (1)  $n_{\text{max}} = p+w$ .

For the PAD detector we can model inefficiencies simply by considering a beam splitter of transitivity  $\eta$  in front of both homodyne detectors [16]. The first observation we make is that for high efficiency, the ideal detector obtains a higher fidelity. The trend with higher photon number is similar for both detectors. Where the advantage lies for the PAD is that the efficiency for current homodyne detectors is very high compared with available photon counters.

For a particular  $\Delta$  and  $\eta$  we can consider an equivalent ideal detector that gives the same fidelity. Constructing an equivalence in this fashion is particularly useful and was considered by Ref. [17], where they compared an ideal photon counter with homodyne detection in the context of quantum communication. As such, they used the mutual information as a means of comparison. For our scheme, we envision state preparation as the main application so we will use the fidelity as a means of comparison. This comparison is plotted in Fig. 7 for the ability to project out the state  $|a_1\rangle$  from the input state  $\sum_{n=0}^4 |a_n\rangle|n\rangle$ . A detector able to achieve this projection forms a *selective* detector which is needed in many linear optics schemes.

### VI. DISCUSSION AND CONCLUSIONS

Because of its nondeterministic nature, we envision applications of this detector mainly in state preparation, where

nonclassical states are prepared through conditioning on photon-number detection. We could prepare a good approximation to an  $|n\rangle$  photon state required by our detector, by using spontaneous parametric down-conversion and a detector cascade in one arm. Even if the detectors in the cascade are inefficient, if, say three detectors register a click, then we have at least a three photon term in the other arm. The errors caused by having more than the required number of photons are offset by the low probability of such events. One intriguing possibility is to employ this detector in a proposal by Dakna *et al.* [18]. In the Dakna scheme, a good approximation to an optical Schrödinger cat state is generated by mixing a single mode squeezed state on a beam splitter with the vacuum and conditioning on detecting a certain number of photons in one of the exit ports.

Another possible extension is to use other parameter choices, and postselection choices to directly project out certain distributions of photon number terms.

We have presented a nondeterministic scheme which functions as a high-fidelity Fock state projector. This detect-

ing scheme allows a tunable trade-off between the fidelity and probability of detection. The weaknesses of the scheme are that it requires an  $|n\rangle$  photon state and that it is nondeterministic. The  $|n\rangle$  photon state could be prepared in the first instance simply by conditioning the output of a spontaneous parametric down converter with a traditional detector cascade. The nondeterministic nature of the scheme leads us to conclude that the main application for the detector will be in state generation.

*Note added.* Recently, a close connection was pointed out between the calculations performed here and the teleportation formalism in Ref. [19].

#### ACKNOWLEDGMENTS

A.G. acknowledges support from the New Zealand Foundation for Research, Science and Technology under Grant No. UQSL0001. This project was supported by the Australian Research Council.

- 
- [1] E. Knill, R. Laflamme, and G. Milburn, *Nature (London)* **409**, 46 (2001).
  - [2] D. Gottesman and I.L. Chuang, *Nature (London)* **402**, 390 (1999).
  - [3] T.B. Pittman, B.C. Jacobs, and J.D. Franson, *Phys. Rev. A* **64**, 062311 (2001).
  - [4] T.C. Ralph, A.G. White, W.J. Munro, and G.J. Milburn, *Phys. Rev. A* **65**, 012314 (2002).
  - [5] E. Knill, *Phys. Rev. A* **66**, 052306 (2002).
  - [6] T.C. Ralph, W.J. Munro, and G.J. Milburn, e-print, quant-ph/0110115.
  - [7] H. Lee, P. Kok, N.J. Cerf, and J.P. Dowling, *Phys. Rev. A* **65**, 030101(R) (2002).
  - [8] P. Kok, H. Lee, and J.P. Dowling, *Phys. Rev. A* **65**, 052104 (2002).
  - [9] J. Fiurásek, *Phys. Rev. A* **65**, 053818 (2002).
  - [10] X. Zou, K. Pahlke, and W. Mathis, *Phys. Rev. A* **66**, 014102 (2002).
  - [11] J. Kim, S. Takeuchi, Y. Yamamoto, and H.H. Hogue, *Appl. Phys. Lett.* **74**, 902 (1999).
  - [12] S. Takeuchi, J. Kim, Y. Yamamoto, and H.H. Hogue, *Appl. Phys. Lett.* **74**, 1063 (1999).
  - [13] P. Kok and S.L. Braunstein, *Phys. Rev. A* **63**, 033812 (2001).
  - [14] E.S. Polzik, J. Carri, and H.J. Kimble, *Phys. Rev. Lett.* **68**, 3020 (1992).
  - [15] S.D. Bartlett, E. Diamanti, B.C. Sanders, and Y. Yamamoto, e-print quant-ph/0204073.
  - [16] U. Leonhardt and H. Paul, *Phys. Rev. A* **48**, 4598 (1993).
  - [17] K. Nemoto and S.L. Braunstein, *Phys. Rev. A* **66**, 032306 (2002).
  - [18] M. Dakna *et al.*, *Phys. Rev. A* **55**, 3184 (1997).
  - [19] H.F. Hofmann *et al.*, *Phys. Rev. A* **62**, 062304 (2000).