PREDICTOR-CORRECTOR METHODS OF RUNGE–KUTTA TYPE FOR STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this paper we construct predictor-corrector (PC) methods based on the trivial predictor and stochastic implicit Runge–Kutta (RK) correctors for solving stochastic differential equations. Using the colored rooted tree theory and stochastic B-series, the order condition theorem is derived for constructing stochastic RK methods based on PC implementations. We also present detailed order conditions of the PC methods using stochastic implicit RK correctors with strong global order 1.0 and 1.5. A two-stage implicit RK method with strong global order 1.0 and a four-stage implicit RK method with strong global order 1.5 used as the correctors are constructed in this paper. The mean-square stability properties and numerical results of the PC methods based on these two implicit RK correctors are reported.

Key words. stochastic differential equations, predictor-corrector methods, Runge–Kutta methods, numerical stability

AMS subject classifications. 60H10, 65L06, 65L20

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1. Introduction. Runge–Kutta (RK) methods are one of the most efficient classes of methods for solving ordinary differential equations (ODEs). Certain classes of implicit RK methods have excellent stability properties and are widely used to solve stiff ODEs. In the last decade, predictor-corrector (PC) methods have been one of the major classes of methods for solving nonstiff ODEs on parallel computers (see [2], [3], [4], [5], [6], [10], [15], [25], [27], and [28]).

By comparing the Taylor series expansion of the approximation solution to the Taylor series expansion of the exact solution over one step assuming exact initial values, Butcher [13] introduced the rooted tree theory that is the key to constructing RK methods for ODEs. As the RK-type PC methods can be represented as a special class of block explicit RK methods, the rooted tree theory has been applied to RK-type PC methods. Burrage [2], [6] has developed a comprehensive theory based on the use of Butcher series which allows the analysis of the local error of any RK-type PC method and has also applied this theory to an analysis of the local behavior of two classes of PC methods, including one which is based on the trivial predictor and an implicit RK corrector.

For solving stochastic differential equations (SDEs), stochastic RK methods are an important class of numerical methods. Rümelin [22] introduced the use of traditional RK methods for SDEs. These methods resemble in their structure deterministic RK methods for ODEs. Burrage and Burrage [7], [8] and Burrage [12] established the colored rooted tree theory and stochastic B-series which is generalized from the corresponding rooted tree theory and B-series for constructing numerical methods for ODEs. Based on these theories, Burrage and Burrage present order conditions for constructing a general class of stochastic RK methods for solving Stratonovich SDEs and also construct an explicit strong global order 1.0 two-stage RK method with...
minimum principal error constants [17] and an explicit five-stage RK method with
strong global order 1.5 [8]. Tian and Burrage [26] consider diagonally semi-implicit
and implicit strong order 1.0 two-stage RK methods with good stability properties or
good accuracy. In addition, in order to avoid the unboundedness of numerical solutions
of the implicit stochastic RK methods, composite RK methods are constructed which
are a combination of semi-implicit RK methods and implicit RK methods. Further
research has been done by Komori, Mitsui, and Sugiural [18], in which they use the
tree expansions of the true and numerical solutions to construct ROW-type schemes
for SDEs.

For solving SDEs, the PC technique has been already applied to linear multistep
implicit methods [1]. For weak solutions of SDEs, Kloeden and Platen [17] and Platen
[21] consider families of PC methods with weak order 1.0 and 2.0. In this paper we
consider PC methods using stochastic RK methods as correctors for strong solutions
of SDEs. In section 2, we first give a brief review of the rooted tree theory for
constructing RK methods and RK-type PC methods for ODEs and then give order
conditions for constructing stochastic RK-type PC methods after a brief review of
the colored rooted tree theory for constructing stochastic RK methods. In section
3, we give the detailed order conditions for two-stage RK-type PC methods with
strong global order 1.0 and then construct a two-stage implicit RK method with
strong global order 1.0. Similar work is done for four-stage RK-type PC methods
with strong global order 1.5 in section 4. The mean-square stability properties of the
RK-type PC methods using these two-stage and four-stage implicit RK correctors are
considered in section 5. Numerical results are reported in section 6.

2. Order conditions for RK-type PC methods. In this section, a brief
review is first given for the rooted tree theory and order conditions for constructing
RK-type PC methods for ODEs. For solving the ODE
\[ y'(t) = f(y(t)), \quad y(t_0) = y_0, \quad t \in [t_0, T], \quad y \in \mathbb{R}^m, \]
the class of s-stage RK methods is given by
\[ Y_i = y_n + h \sum_{j=1}^{s} a_{ij} f(Y_j), \quad i = 1, 2, \ldots, s, \]
\[ y_{n+1} = y_n + h \sum_{j=1}^{s} b_j f(Y_j), \]
(2.1)
which can be represented by the so-called Butcher tableau
\[ \begin{array}{c|c}
   \ c \\
   A \\
   b^T
\end{array}, \quad c = Ae, \quad e = (1, \ldots, 1)^T \in \mathbb{R}^s. \]

In order to express derivatives of \( f(y) \) systematically, Butcher [13] introduced
the rooted tree theory which provides a general framework for studying order conditions
of RK methods. Let \( T \) be the set of rooted trees and \( t = [t_1, \ldots, t_m] \) be the tree
formed by joining subtrees \( t_1, \ldots, t_m \) each by a single branch to a common root. In
addition, let \( \phi \) denote the empty tree and \( \tau \) the unique tree with one node. For each
\( t \), denote \( \rho(t) \) as the number of nodes (vertices) of \( t \), \( h(t) \) as the height of \( t \), with the
height of the unique tree \( \tau \) being 1, respectively. Then the elementary differential
associated with \( t = [t_1, \ldots, t_m] \) is given by
\[ F(t)y = f^{(m)}(F(t_1)y, \ldots, F(t_m)y), \quad F(\phi) = y. \]
With these definitions, the following order theorem holds for RK methods (see Burrage [6]).

**Theorem 2.1.** A RK method is of order \(w\) if and only if

\[ e(t) = 0 \quad \forall \rho(t) \leq w, \]

where for any tree \(t = [t_1, \ldots, t_m]\)

\[ e(\phi) = 0, \quad e(t) = 1 - \rho(t)b^\top \prod_{j=1}^m k(t_j), \]

with

\[ k(\phi) = e, \quad k(t) = \rho(t) \prod_{j=1}^m (Ak(t_j)). \]

Now consider a RK-type PC method which uses a RK corrector (2.1) and the trivial predictor based on the update value \(y_n\), given by

\[
Y^{(0)} = (e \otimes I)y_n,
\]

\[
Y^{(k)} = (e \otimes I)y_n + h(A \otimes I)f(Y^{(k-1)}), \quad k = 1, 2, \ldots, l,
\]

\[
y_{n+1} = y_n + hb^\top f(Y^{(l)}),
\]

where \(Y = (Y_1^\top, \ldots, Y_s^\top)^\top\) and \(f(Y) = (f(Y_1^\top, \ldots, f(Y_s^\top))^\top\). This method can be represented by a \((l+1)s\)-stage explicit RK method, whose Butcher tableau is given by

\[
\begin{array}{cccc}
0 & 0 \\
c & A & 0 \\
c & 0 & A & 0 \\
\vdots & \ddots & \ddots & \ddots \\
c & 0 & \cdots & 0 & A & 0 \\
0 & \cdots & 0 & 0 & b^\top \\
\end{array}
\]

Applying the order conditions for RK methods (Theorem 2.1) to this \((l+1)s\)-stage explicit RK method, Burrage [2], [3], [6] presents a theoretical tool for measuring the error behavior of this RK-type PC method and gives the order conditions of this method.

**Theorem 2.2.** If a RK corrector is applied to the trivial predictor with \(l\) corrections, then the local error is given by

\[
l_{n+1} = \sum_{t \in T^*} e(t) [F(t)] y(t_n) \frac{h(\rho(t))}{\rho(t)!},
\]

where for \(t = [t_1, \ldots, t_m]\)

\[
e(t) = 1 - \rho(t)b^\top \prod_{i=1}^m k(t_i),
\]

\[ (2.3) \]
with

\[ k_0(\phi) = e, \quad k_0(t) = 0, \quad \rho(t) > 0, \]
\[ k_{j+1}(t) = \rho(t) \prod_{i=1}^{m} (Ak_j(t_i)), \quad j = 0, 1, \ldots, l - 1. \]

By studying the behavior of the local errors of a RK-type PC method, Burrage [2], [6] has shown that each application of a corrector increases the order of the overall method by one until the order of the corrector is reached. In addition, when the number of corrections is such that the order cannot increase further, then the effect of more corrections is to shift the errors due to the predictor further away from the principal error terms.

Now we consider the order conditions of stochastic RK-type PC methods for the Stratonovich SDE driven by principal error terms.

\[ dy(t) = g_0(y(t)) dt + \sum_{j=1}^{d} g_j(y(t)) \circ dW_j(t), \quad y(t_0) = y_0, \quad y \in \mathbb{R}^m, \]
where the deterministic term \( g_0(y(t)) \) is the drift coefficient, the stochastic term \( g_j(y(t)) \) \((j = 1, \ldots, d)\) are the diffusion coefficients, and \( W_j(t) \) is the Wiener process, whose increment \( \Delta W_j(t) = W_j(t + \Delta t) - W_j(t) \) is a Gaussian random variable \( N(0, \Delta t) \).

The solution of (2.5) can be written in integral form as

\[ y(t) = y(t_0) + \int_{t_0}^{t} g_0(y(t)) dt + \sum_{j=1}^{d} \int_{t_0}^{t} g_j(y(t)) \circ dW_j(t), \]
and can also be expressed as a stochastic Taylor series, given by

\[ y(t) = y_0 + \sum_{j_1=0}^{d} g_{j_1}(y_0)J_{j_1,t} + \sum_{j_1,j_2=0}^{d} L^{j_1} g_{j_2}(y_0)J_{j_1,j_2,t} \]
\[ + \sum_{j_1,j_2,j_3=0}^{d} L^{j_1} L^{j_2} g_{j_3}(y_0)J_{j_1,j_2,j_3,t} + \cdots, \]
where the Stratonovich operator is defined by

\[ L^j = \sum_{k=1}^{m} g^k_j \frac{\partial}{\partial y^k}, \quad j = 0, 1, \ldots, d \]
and \( J_{j_1,\ldots,j_k,t} \) represents the Stratonovich multiple integral which is defined recursively by (see [16] and [17])

\[ J_{0,t} = \int_{t_0}^{t} dt = t - t_0, \]
\[ J_{j,t} = \int_{t_0}^{t} \circ dW_j(t) = \Delta W_j(t), \]
\[ J_{j_1,j_2,\ldots,j_k-1,j_k,t} = \int_{t_0}^{t} J_{j_1,j_2,\ldots,j_k-1,t} \circ dW_j(t), \quad j_k = 0, \]
\[ J_{j_1,j_2,\ldots,j_k-1,j_k,t} = \int_{t_0}^{t} J_{j_1,j_2,\ldots,j_k-1,t} \circ dW_j(t), \quad j_k = j, \quad j = 1, \ldots, d. \]
In order to express the stochastic Taylor series more precisely, Burrage and Burrage present the colored rooted tree theory [7] and stochastic B-series [8] which have the same structure as the corresponding rooted tree theory and B-series.

**Definition 2.3.** The \((d+1)\)-colored rooted trees can be defined recursively by

(i) the elementary rooted tree is \(\tau_k\) which represent the deterministic elementary rooted tree \(\tau_0\) if \(k = 0\) and a stochastic one \(\tau_k\) if \(k \in \{1, 2, \ldots, d\}\);

(ii) if \(t_1, \ldots, t_m\) are \((d+1)\)-colored rooted trees, then \([t_1, \ldots, t_m]_k\) is the \((d+1)\)-colored rooted tree in which \(t_1, \ldots, t_m\) are each joined by a single branch to \(\tau_k\) \((k \in \{1, 2, \ldots, d\})\).

Similar to the rooted tree theory for ODEs, denote \(T_1\) as the set of all \((d+1)\)-colored rooted trees, \(\rho(t)\) as the number of vertices of \(t\), \(\alpha(t)\) as the number of ways of labelling the vertices of \(t\) so that the labels increase outwardly along the arcs, \(h(t)\) as the height of \(t\) where the height of the elementary tree is 1, and \(\gamma(t)\) as the density of \(t = [t_1, \ldots, t_m]_k\), defined by

\[
\gamma(t) = \rho(t) \prod_{j=1}^{m} \gamma(t_j)
\]

and where \(J(t)\) represents the corresponding \(J\)-integral associated with tree \(t\) which is defined by

\[
J(t)(h) = \int_{0}^{h} \prod_{j=1}^{m} J(t_j)(s) \circ dW_k(s), \quad J(\tau_k)(h) = W_k(h).
\]

In a similar manner to the deterministic case, an elementary differential can be associated with any \(t \in T_1\) such that

\[
F(\tau_k)(y) = g_k(y), \\
F(t)(y) = g_k^{(m)}(y)[F(t_1)(y), \ldots, F(t_m)(y)], \quad t = [t_1, \ldots, t_m]_k.
\]

With the definitions of \((d+1)\)-colored rooted trees, Burrage and Burrage [7] and Burrage [12] have given the Taylor series expansion of the exact solution of an SDE.

**Theorem 2.4.** The Stratonovich–Taylor series for the actual solution of the SDE given by (2.5) (together with initial value \(y(t_0) = y_0\)) is

\[
y(t_0 + h) = \sum_{t \in T_1} \frac{\gamma(t)}{\rho(t)!} J(t)\alpha(t) F(t)(y(t_0)),
\]

where \(F(t)(y)\) is the elementary differential defined by the structure of tree \(t\), and \(J(t)\) represents the corresponding \(J\)-integral associated with tree \(t\).

For solving the SDE (2.5), a general class of \(s\)-stage stochastic RK method derived by Burrage and Burrage [7] and Burrage [12] is given by

\[
Y_i = y_n + \sum_{k=0}^{d} \sum_{j=1}^{s} z_{ij}^{(k)} g_k(Y_j), \quad i = 1, \ldots, s,
\]

\[
y_{n+1} = y_n + \sum_{k=0}^{d} \sum_{j=1}^{s} z_{j}^{(k)} g_k(Y_j),
\]

(2.7)
where $Z_{ij}^{(k)}$ and $z_{j}^{(k)}$ are random variables, which are functions of $h$, to be determined based on order and stability analysis. Note that in the case of the deterministic parameters $Z^{(0)}$ and $z^{(0)}$, $h$ is included implicitly in these terms.

The numerical solution obtained by the stochastic RK method (2.7) can be written as a Taylor series expansion [7], given by

$$y(t_0 + h) = \sum_{t \in T} \frac{\gamma(t)}{\rho(t)!} a(t) \alpha(t) F(t)(y(t_0)),$$

where, for $t = [t_1, \ldots, t_m]$, $a(t)$ is defined by

$$a(t) = z^{(k)^T} \Phi(t),$$
$$\Phi(t) = \prod_{i=1}^{m} (Z^{(k)} \Phi(t_i)), \quad \Phi(t_0) = e.$$

In designing numerical schemes for solving SDEs, some criteria are needed to measure the efficiency of a numerical scheme by means of its order of convergence. There are two criteria to measure the convergence order: strong convergence and weak convergence. For problems involving direct simulations of paths, it is required that the simulated sample paths be close to the exact solution of the original SDE. This consideration leads to the strong convergence criterion (for example, see Burrage [12]).

**Definition 2.5.** Let $y_N$ be the numerical approximation to $y(t_N)$ at time $T = Nh + t_0$ after $N$ steps with constant stepsize $h$; then $y$ is said to converge strongly to $y$ with order $p$ if there exist $C > 0$ (independent of $h$ but dependent on the length of the time interval $T - t_0$) and $\delta > 0$ such that

$$E(|y_N - y(t_N)|) \leq Ch^p, \quad h \in (0, \delta).$$

The local truncation error at $t = t_{n+1}$ of the stochastic RK method (2.7) can be written as

$$L_n = \sum_{t \in T} \frac{\gamma(t)}{\rho(t)!} \alpha(t) (J(t) - a(t)) F(t)(y(t_n)).$$

Burrage and Burrage [8] have given the following definition to measure the accuracy of the RK methods

**Definition 2.6.** This stochastic RK method will have strong local order $p$ if

$$E[|L_n|] = O(h^{p + \frac{1}{2}})$$

and will have mean local order $p$ if

$$E(L_n) = O(h^{p+1}).$$

In addition they have proven the following theorem concerning the relationship between the local error behavior and the global error behavior (see also Milstein [19]), given by the following theorem.

**Theorem 2.7.** Let the $g_j$ possess all necessary partial derivatives for all $y \in \mathbb{R}^m$; then if

$$(E[|l_n|^2])^{1/2} = O(h^{p + 1/2}) \quad \forall n$$
and
\[ E[t_n] = O(h^{p+1}) \quad \forall n, \]
then
\[ (E[\|\epsilon_N\|^2])^{1/2} = O(h^p), \]
where \( \epsilon_N \) is the global error at step point \( t_N \) with the assumption of the exact initial solution of \( y_0 = y(t_0) \).

Thus the stochastic RK method (2.7) is of strong global order \( p \) if it has strong local order \( p \) and mean local order \( p \).

For solving the SDE (2.5), the stochastic RK-type PC method, which is based on a stochastic RK corrector (2.7) and the trivial predictor, is given by
\[
Y^{(0)} = (e \otimes I)y_n, \\
Y^{(i)} = (e \otimes I)y_n + \sum_{k=0}^{d} (Z^{(k)} \otimes I)g_k(Y^{(i-1)}), \quad i = 1, 2, \ldots, l,
\]
\[
y_{n+1} = y_n + \sum_{k=0}^{d} (z^{(k)} \otimes I)g_k(Y^{(l)}),
\]
where \( Y^{(i)} = (Y^{(i)}_1, \ldots, Y^{(i)}_s)^\top, Z^{(k)} = (Z^{(k)}_{ij})_{s \times s}, \) and \( z^{(k)} = (z^{(k)}_1, \ldots, z^{(k)}_s), (k = 0, 1, \ldots, d) \). This stochastic RK-type PC method can be represented by an \((l+1)s\)-stage block explicit stochastic RK method characterized by the tableau
\[
\begin{array}{ccccccc}
0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\
0 & Z^{(0)} & 0 & \cdots & \cdots & Z^{(d)} & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & Z^{(0)} & 0 & \cdots & 0 \\
0 & \cdots & 0 & z^{(0)} & \cdots & 0 & z^{(d)} \\
\end{array}
\]

Applying the order theorem for stochastic RK methods to (2.8), we have the main theorem on order conditions for constructing stochastic RK-type PC methods (2.8) in this paper.

**Theorem 2.8.** If a stochastic RK corrector is applied to the trivial predictor with \( l \) corrections, then the strong local error of the stochastic RK-type PC method is given by
\[
l_{n+1} = \sum_{t \in T_1} \gamma(t) \frac{c(t)\alpha(t)}{\rho(t)} \[F(t)]y(t_n),
\]
where for \( t = [t_1, \ldots, t_m] \), \( e(t) = J(t) - a(t) \) and \( a(t) \) is given by
\[
a(t) = z^{(k)}_1 \Phi_1(t),
\]
and
\[
\Phi_0(t_k) = e, \quad \Phi_0(t) = 0, \quad \rho(t) \geq 2,
\]
\[
\Phi_{j+1}(t) = \prod_{i=1}^{m} (Z^{(k)} \Phi_j(t_i)).
\]
As a special case the expressions of $a_i(t)$ are considered with a different number of corrections. When no correction is performed ($l = 0$), then

\begin{align*}
    t_1 &= \tau_j, \quad a_0(t_1) = z^{(j)}e, \\
    t_2 &= \tau_0, \quad a_0(t_2) = z^{(0)}e, \\
    h(t) &\geq 2, \quad a_0(t) = 0.
\end{align*}

Here for $t_1$, $j = 1, 2, \ldots, d$. This notation is also valid for trees $t_3 \sim t_{18}$ in the following discussion.

If one correction is performed ($l = 1$), the expressions for $a_1(t)$ associated with trees $t_1$ and $t_2$ are the same as the corresponding $a_0(t)$, namely $a_1(t_i) = a_0(t_i)$ ($i = 1, 2$). For trees with more vertices, then, assuming that the $j_i$ are nonzero,

\begin{align*}
    t_3 &= [\tau_{j_1}]_{j_2}, \quad a_1(t_3) = z^{(j_2)}Z^{(j_1)}e, \\
    t_4 &= [\tau_0]_{j_1}, \quad a_1(t_4) = z^{(j_1)}Z^{(0)}e, \\
    t_5 &= [\tau_{j_1}]_0, \quad a_1(t_5) = z^{(0)}Z^{(j_1)}e, \\
    t_6 &= [\tau_0]_0, \quad a_1(t_6) = z^{(0)}Z^{(0)}e, \\
    t_7 &= [\tau_{j_1}, \tau_{j_2}]_{j_3}, \quad a_1(t_7) = z^{(j_3)}(Z^{(j_1)}e)(Z^{(j_2)}e), \\
    t_8 &= [\tau_{j_1}, \tau_{j_2}]_0, \quad a_1(t_8) = z^{(0)}(Z^{(j_1)}e)(Z^{(j_2)}e), \\
    t_9 &= [\tau_{j_1}, \tau_{j_2}]_{j_3}, \quad a_1(t_9) = z^{(j_2)}(Z^{(j_1)}e)(Z^{(0)}e), \\
    t_{10} &= [\tau_0, \tau_{j_1}]_{j_2}, \quad a_1(t_{10}) = z^{(j_2)}(Z^{(0)}e)(Z^{(j_1)}e), \\
    t_{11} &= [\tau_{j_1}, \tau_{j_2}, \tau_{j_3}]_{j_4}, \quad a_1(t_{11}) = z^{(j_4)}(Z^{(j_1)}e)(Z^{(j_2)}e)(Z^{(j_3)}e), \\
    h(t) &\geq 3, \quad a_1(t) = 0.
\end{align*}

If two corrections are performed ($l = 2$), the expressions for $a_2(t)$ associated with trees $t_1, \ldots, t_{11}$ are the same as the corresponding $a_1(t)$, namely $a_2(t_i) = a_1(t_i)$ ($i = 1, \ldots, 11$). For trees with more vertices, then

\begin{align*}
    t_{12} &= [[\tau_{j_1}]_{j_2}]_{j_3}, \quad a_2(t_{12}) = z^{(j_3)}Z^{(j_2)}Z^{(j_1)}e, \\
    t_{13} &= [[\tau_0]_{j_1}]_{j_2}, \quad a_2(t_{13}) = z^{(j_2)}Z^{(j_1)}Z^{(0)}e, \\
    t_{14} &= [[\tau_{j_1}]_0]_{j_2}, \quad a_2(t_{14}) = z^{(j_2)}Z^{(0)}Z^{(j_1)}e, \\
    t_{15} &= [[[\tau_{j_1}]]_{j_2}]_0, \quad a_2(t_{15}) = z^{(0)}Z^{(j_2)}Z^{(j_1)}e, \\
    t_{16} &= [[\tau_{j_1}, \tau_{j_2}]_{j_3}]_{j_4}, \quad a_2(t_{16}) = z^{(j_4)}(Z^{(j_2)}Z^{(j_1)}e)(Z^{(j_3)}e), \\
    t_{17} &= [[\tau_{j_1}, \tau_{j_2}, \tau_{j_3}]_{j_4}, \quad a_2(t_{17}) = z^{(j_4)}Z^{(j_3)}((Z^{(j_1)}e)(Z^{(j_2)}e)), \\
    h(t) &\geq 4, \quad a_2(t) = 0.
\end{align*}

When a stochastic RK-type PC method is corrected three times, the expressions for $a_3(t)$ associated with trees $t_i$ ($i = 1, \ldots, 17$) are the same as the corresponding $a_2(t)$, namely

\begin{align*}
    a_3(t_i) = a_2(t_i), \quad i = 1, \ldots, 17.
\end{align*}

For the analysis of the stochastic RK-type PC methods in this paper, we need only consider additionally the expression $a_3(t)$ for the tree $[[[\tau_{j_1}]_{j_2}]_{j_3}]_{j_4}$, where none of the $j_i$ is zero, given by

\begin{align*}
    t_{18} &= [[[\tau_{j_1}]_{j_2}]_{j_3}]_{j_4}, \quad a_3(t_{18}) = z^{(j_4)}Z^{(j_3)}Z^{(j_2)}Z^{(j_1)}e.
\end{align*}

In the following sections the order conditions associated with trees $t_1, \ldots, t_{18}$ are used to construct stochastic RK-type PC methods.
3. Strong order 1.0 RK methods. The order theory developed in section 2 will apply to the very general class of problems (2.5) with \( d > 1 \). However, due to spatial constraints and the extreme difficulty in solving the order conditions for the arbitrary \( d \) case, we will focus on constructing effective PC methods for \( d = 1 \).

The \( s \)-stage RK methods with one stochastic variable \( J_1 \sim N(0, h) \) are given by
\[
Y = (e \otimes I)y_n + h(A \otimes I)g_0(Y) + J_1(B \otimes I)g_1(Y),
\]
\[
y_{n+1} = y_n + h(\alpha^T \otimes I)g_0(Y) + J_1(\beta^T \otimes I)g_1(Y),
\]
where \( A \) and \( B \) are \( s \times s \) matrices, while \( \alpha \) and \( \beta \) are \( s \)-dimensional vectors. According to the theorems given by Rümelin [22] and Burrage, Burrage, and Belward [9], the maximum strong global order of these stochastic RK methods is 1.0.

For the trivial predictor, the stochastic RK-type PC method using (3.1) as the corrector is given by
\[
Y^{(0)} = (e \otimes I)y_n, \quad Y^{(i)} = (e \otimes I)y_n + h(A \otimes I)g_0(Y^{(i-1)}) + J_1(B \otimes I)g_1(Y^{(i-1)}), \quad i = 1, \ldots, l, \\
y_{n+1} = y_n + h(\alpha^T \otimes I)g_0(Y^{(l)}) + J_1(\beta^T \otimes I)g_1(Y^{(l)}).
\]

Now consider the order conditions of the RK-type PC method (3.2). If no correction is performed, the local truncation error of this method is given by
\[
l_{10} = h(1 - \alpha^T e)F(\tau_0)(y(t_n)) + J_1(1 - \beta^T e)F(\tau_1)(y(t_n)) + \sum_{\rho(t) \geq 2} J(t)F(t)(y(t_n)).
\]
Assuming that
\[
\alpha^T e = 1, \quad \beta^T e = 1,
\]
this method will have strong local order 0.5, namely \( E(l_{10}^2) = O(h^2) \). In this case the PC method (3.2) is equivalent in strong order to the Euler–Maruyama method, given by
\[
y_{n+1} = y_n + hg_0(y_n) + J_1 g_1(y_n).
\]
It is well known that the numerical solution of the Euler–Maruyama method converges to the exact solution of the corresponding Itô SDE. Thus the numerical solution of method (3.2) without any correction may not converge to the exact solution of the Stratonovich SDE (2.5).

If one correction is performed (\( l = 1 \)), method (3.2) will have strong local order 1.0 if, in addition to (3.3),
\[
e(t_3) = J(t_3) - a(t_3) = \left( \frac{1}{2} - \beta^T Be \right) J_1^2 = 0,
\]
which is equivalent to
\[
\beta^T Be = \frac{1}{2}.
\]
At the same time this method will have mean local error 1.0 as
\[ E(e(t_i)) = 0, \quad i = 4, 5, 7, 12, \]
where trees \( t_4, t_5, t_7, \) and \( t_{12} \) are those associated with terms corresponding to \( h^{1.5} \). Thus the stochastic RK-type PC method (3.2) will have strong global order 1.0 if one correction is applied and the order conditions (3.3) and (3.4) are satisfied at the same time.

The order conditions (3.3) and (3.4) of the stochastic RK-type PC method with strong global order 1.0 are the same as those of the stochastic RK methods (3.1) with strong global order 1.0, given in [12]. Thus the strong global order of the RK-type PC method (3.2) is 1.0 if the strong global order of the original stochastic RK method (3.1) is 1.0 and one correction is applied.

Now we construct a two-stage implicit RK method. As there are only three order conditions in (3.3) and (3.4) and 12 coefficients in this method, additional conditions can be considered. For example, we can consider the stochastic order conditions on which the terms corresponding to \( h_{10} \) have minimum coefficients, namely the stochastic order conditions for minimum principal error coefficients. The principal error coefficients are minimized if [12]
\[ \alpha^\top Be = \frac{1}{2}, \quad \beta^\top Ae = \frac{1}{2}, \quad \beta^\top (Be)^2 = \frac{1}{3}, \quad \beta^\top B(Be) = \frac{1}{6}. \]

These four conditions are called the minimum principal error conditions.

Combining the order conditions (3.3) and (3.4) and the minimum principal error conditions together and assuming that \( A = B \) and \( \alpha = \beta \), we have the following two-stage implicit RK corrector method with strong global order 1.0, called IRK2, given by

\[
\begin{pmatrix}
\frac{1}{3} & \frac{1 - \sqrt{3}}{6} & \frac{1}{3} & \frac{1 - \sqrt{3}}{6} \\
\frac{1 + \sqrt{3}}{6} & \frac{1}{3} & \frac{1 + \sqrt{3}}{6} & \frac{1}{3} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}.
\]

4. Strong order 1.5 RK methods. The second special class of the stochastic RK methods (2.7) that will be discussed is those with two stochastic variables \( J_1 \) and \( J_{10}/h \), for solving problems of the form (2.5) with \( d = 1 \), given by

\[
Y = (e \otimes I) g_n + h(A \otimes I) g_0(Y) + \left( J_1(B_1 \otimes I) + \frac{J_{10}}{h} (B_2 \otimes I) \right) g_1(Y),
\]
\[
y_{n+1} = y_n + h(\alpha^\top \otimes I) g_0(Y) + \left( J_1(\beta_1^\top \otimes I) + \frac{J_{10}}{h} (\beta_2^\top \otimes I) \right) g_1(Y),
\]
where \( A, B_1, \) and \( B_2 \) are \( s \times s \) matrices and \( \alpha, \beta_1, \) and \( \beta_2 \) are \( s \)-dimensional vectors. Here we remind readers that on the interval \( [t_n, t_{n+1}] \), \( J_1 \) and \( J_{10}/h \) are closely related. In particular, if \( u \) and \( v \) are two independent \( N(0, 1) \) random variables, then
\[ J_1 = u \sqrt{h}, \quad \frac{J_{10}}{h} = \frac{\sqrt{h}}{2} \left( u + \frac{v}{\sqrt{3}} \right). \]

Burrage and Burrage [7], [8] and Burrage [12] first present this class of stochastic RK methods and study the order conditions of these methods based on the colored
rooted tree theory and stochastic B-series. A five-stage explicit stochastic RK method with strong global order 1.5 is presented in [8].

For the trivial predictor, the stochastic RK-type PC method using (4.1) as a corrector is given by

\begin{align*}
Y(0) = (e \otimes I)y_n, \\
Y(i) = (e \otimes I)y_n + h(A \otimes I)g_0(Y^{(i-1)}) + \left(J_1(B_1 \otimes I) + \frac{J_{10}}{h}(B_2 \otimes I)\right)g_1(Y^{(i-1)}),
\end{align*}

\[i = 1, \ldots, l,\]

\[y_{n+1} = y_n + h(\alpha^\top \otimes I)g_0(Y(l)) + \left(J_1(B_1 \otimes I) + \frac{J_{10}}{h}(B_2 \otimes I)\right)g_1(Y(l)).\]

Now consider the order conditions for this RK-type PC method. When one correction is performed, the order conditions necessary for strong order 1.5 associated with trees \(t_1, \ldots, t_{11}\) are given by

\[E(e^2(t_i)) = 0, \quad i = 1, \ldots, 11.\]

Let \(c = Ae, b = B_1e,\) and \(d = B_2e;\) the above order conditions are equivalent to (see [7] and [12])

\begin{align*}
\alpha^\top(c, d, b) &= \left(1, \frac{1}{2}, 1, 0\right), \\
\beta_1^\top(c, b, d, b^2, d^2) &= \left(1, 1, \frac{1}{2}, -\beta_2^\top b, \frac{1}{3}, -2\beta_2^\top bd\right), \\
\beta_2^\top(c, c, d, b^2, d^2) &= (0, -1, 0, -2\beta_1^\top bd, 0).
\end{align*}

When method (4.2) is corrected twice, the order condition associated with tree \(t_{12}\) is \(E(e^2(t_{12})) = 0,\) which is equivalent to (see [7] and [12])

\begin{align*}
\beta_1^\top B_1b &= \frac{1}{6}, \quad \beta_2^\top B_1b + \beta_1^\top(B_2b + B_1d) = 0, \\
\beta_2^\top B_2d &= 0, \quad \beta_1^\top B_2d + \beta_2^\top(B_2b + B_1d) = 0.
\end{align*}

In order to get mean local order 1.5, it is necessary that the following mean order conditions should be satisfied:

\[E(e(t_i)) = 0, \quad i = 8, 9, 11, 13, 14, 15, 16, 17,\]

which are equivalent to (see [8])

\begin{align*}
0 &= \alpha^\top B_1b + \frac{1}{2}\alpha^\top(B_1d + B_2b) + \frac{1}{3}\alpha^\top B_2d, \\
0 &= \alpha^\top(b^2 + bd + \frac{1}{3}d^2), \\
0 &= \beta_1^\top Ab + \frac{1}{2}(\beta_1^\top Ad + \beta_2^\top Ab) + \frac{1}{3}\beta_2^\top Ad, \\
0 &= \beta_1^\top\left(cb + \frac{1}{2}cd\right) + \beta_2^\top\left(\frac{1}{2}cb + \frac{1}{3}cd\right),
\end{align*}
additional order conditions are considered here, given by method (4.1) is 1.5.

method (4.2) with three corrections is 1.5 if the strong global order of the original RK
strong global order 1.5, given in [8]. Thus the strong global order of the RK-type PC
(4.7)

strong global order 1.5 RK corrector method, which is called IRK4, with matrices

B

(4.5) is needed. When a third correction is performed, the mean order condition associated

In order to get a RK-type PC method with strong global order 1.5, a third correction

(4.6) are satisfied, the strong local order of the RK-type PC method (4.2) is

However, when two corrections are performed and all of the order conditions

∼

1.00436335789 −0.56006282797 −0.41253045082 −0.03177007950
−0.04300768840 −0.12902306500 −0.04300768833 −0.04300768833
2.26132980150 −2.30000000000 0.11987418760 −0.33925011871
3.51937593150 −2.30000000000 0.11987418760 −0.33925011871

It should be noticed that, for expectation in the mean, trees \( t_0 \) and \( t_{10} \) are equivalent.

However, when two corrections are performed and all of the order conditions

(4.3)–(4.5) are satisfied, the strong local order of the RK-type PC method (4.2) is

1.5, but the mean local order of this method is still 1.0 as the mean order condition
associated with tree \( t_{18} \) is not satisfied, since the height of \( t_{18} \) is 4 and so \( a_2(t_{18}) = 0 \).

In order to get a RK-type PC method with strong global order 1.5, a third correction

is needed. When a third correction is performed, the mean order condition associated
with tree \( t_{18} \) is given by \( E(e(t_{18})) = 0 \), which is equivalent to [8]

\[
\begin{align*}
\frac{1}{8} &= \beta_1^T \left( B_1 \left( 3b^2 + 3bd + \frac{5}{6}d^2 \right) + B_2 \left( \frac{3}{2}b^2 + \frac{5}{3}bd + \frac{1}{2}d^2 \right) \right) \\
&\quad + \beta_2^T \left( B_1 \left( \frac{3}{2}b^2 + \frac{5}{3}bd + \frac{1}{2}d^2 \right) + B_2 \left( \frac{5}{6}b^2 + bd + \frac{1}{3}d^2 \right) \right),
\end{align*}
\]

(4.6)

The order conditions (4.3)–(4.6) of the stochastic RK-type PC method with
strong global order 1.5 are the same as those of the stochastic RK method (4.1) with
strong global order 1.5, given in [8]. Thus the strong global order of the RK-type PC
method (4.2) with three corrections is 1.5 if the strong global order of the original RK
method (4.1) is 1.5.

Now an implicit four-stage RK method with strong global order 1.5 is constructed.
In order to have small error coefficients for the deterministic terms, the following
additional order conditions are considered here, given by

\[
\begin{align*}
\alpha^T A c &= \frac{1}{6}, \quad \alpha^T A c^2 &= \frac{1}{12}, \quad \alpha^T \text{Diag}(c)Ac = \frac{1}{8}, \quad \alpha^T A^2 c &= \frac{1}{24}.
\end{align*}
\]

(4.7)

Using Maple to solve all of the order conditions (4.3)–(4.7), we have the following
strong global order 1.5 RK corrector method, which is called IRK4, with matrices \( A, \)
\( B_1, \) and \( B_2: \)

\[
A = \begin{pmatrix}
1.00436335789 & -0.56006282797 & -0.41253045082 & -0.03177007950 \\
-0.04300768840 & -0.12902306500 & -0.04300768833 & -0.04300768833 \\
2.26132980150 & -2.30000000000 & 0.11987418760 & -0.33925011871 \\
3.51937593150 & -2.30000000000 & 0.11987418760 & -0.33925011871
\end{pmatrix},
\]

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\[ B^{(1)} = \begin{pmatrix} 0.1056624326 & 0.03522081088 & 0.03522081088 & 0.03522081088 \\ 0.19716878372 & 0.19716878372 & 0.19716878372 & 0.19716878372 \\ -0.19879713087 & 0.53213046420 & 0.50000000000 & 0.16666666667 \\ 0.1666666667 & 0.1666666667 & 0.1666666667 & 0.50000000000 \end{pmatrix}, \]


and weight vectors \( \alpha^\top \), \( \gamma^{(1)} \), and \( \gamma^{(2)} \):

\[
\alpha^\top = (1.205542599, 0.2329045687, -0.7937286771, 0.3552815092),
\gamma^{(1)} = (0.5, 0.5, -0.8974417060, 0.8974417060),
\gamma^{(2)} = (0, 0, 0.7948834118, -0.7948834118).
\]

**Remark.** This method requires only four parallel stages and three sequential stages (cf. the strong order 1.5 explicit stochastic RK method G5 of [8] which requires five sequential stages) and so is implemented efficiently on a four processor computer.

5. **Stability properties of RK-type PC methods.** In this paper the following linear test equation of Stratonovich type, given by

\[ dy = ay dt + by \circ dW(t), \quad y(0) = y_0, \tag{5.1} \]

is used to discuss the stability properties of stochastic RK-type PC methods.

Applying a one-step numerical scheme to (5.1), this numerical scheme is represented by

\[ y_{n+1} = R(h, a, b)y_n. \]

Saito and Mitsui [24] introduced the definition of mean-square (MS) stability.

**Definition 5.1.** A numerical scheme is said to be MS-stable for \( h, a, \) and \( b \) if

\[ \overline{\mathcal{R}}(h, a, b) = E(|R(h, a, b)|^2) < 1. \]

\( \overline{\mathcal{R}}(h, a, b) \) is called the MS-stability function of the numerical scheme.

Another important stability definition is that of asymptotic stability. Saito and Mitsui [23] introduced the definition of T-stability to measure asymptotic stability and give two examples on the T-stability properties of numerical methods for weak solutions. Burrage and Tian [11] present a method to measure the T-stability for strong solutions and give the definition of T(A)-stability. Here we just consider the MS-stability properties of the stochastic RK-type PC methods presented in this paper.

Applying the stochastic RK-type PC methods (2.8) to (5.1) gives

\[
Y^{(0)} = ey_n,
Y^{(i)} = ey_n + aZ^{(0)}Y^{(i-1)} + bZ^{(1)}Y^{(i-1)} \\
= \left[ I + Z + Z^2 + \cdots + Z^i \right] ey_n, \quad i = 1, 2, \ldots, l,
\]

\[ y_{n+1} = y_n + aZ^{(0)}Y^{(l)} + bZ^{(1)}Y^{(l)} \\
= \left( 1 + Z \left[ I + Z + Z^2 + \cdots + Z^l \right] e \right) y_n. \]
where
\[ \bar{z} = az^{(0)} + bz^{(1)}, \quad \bar{Z} = aZ^{(0)} + bZ^{(1)}. \]

Let
\[ R^{(l)} = 1 + \bar{z} \left[ I + \bar{Z} + \bar{Z}^2 + \cdots + \bar{Z}^l \right] e; \]
then the stochastic RK-type PC methods (2.8) are MS-stable for \( h, a, \) and \( b \) if
\[ R^{(l)} = E \left( \left| R^{(l)} \right|^2 \right) < 1. \]

Now we consider the MS-stability properties of the stochastic RK-type PC methods (3.2) with strong global order 1.0. Applying these methods to (5.1) gives
\[ R^{(l)}(p, q, J_1) = 1 + (p\alpha + qJ_1\beta \top) \left( \sum_{i=0}^{l} (pA + qJ_1B)^i \right) e, \]
where \( p = ah, \quad q = b\sqrt{h}, \quad \) and \( J_1 = J_1\sqrt{h} \sim N(0, 1). \) For the stochastic RK-type PC method based on the two-stage implicit RK corrector IRK2 (3.5), the expressions for \( R^{(l)} \) are given by
\[ R^{(1)} = 1 + p + qJ_1 + \frac{1}{2}(p + qJ_1)^2, \quad R^{(2)} = 1 + p + qJ_1 + \frac{1}{2}(p + qJ_1)^2 + \frac{1}{6}(p + qJ_1)^3, \]
\[ R^{(3)} = 1 + p + qJ_1 + \frac{1}{2}(p + qJ_1)^2 + \frac{1}{6}(p + qJ_1)^3 + \frac{1}{36}(p + qJ_1)^4, \]
and the MS-stability functions are given by
\[ R^{(1)}_1 = 1 + 2p + 2p^2 + p^3 + \frac{1}{4}p^4 + 2q^2 + 3pq^2 + \frac{3}{2}p^2q^2 + \frac{3}{4}q^4, \]
\[ R^{(2)}_1 = 1 + 2p + 2p^2 + \frac{4}{3}p^3 + \frac{7}{12}p^4 + \frac{1}{6}p^5 + \frac{1}{36}p^6 + 2q^2 + 4pq^2 + \frac{7}{2}p^2q^2 + \frac{5}{3}p^3q^2 + \frac{7}{4}q^4 + \frac{5}{2}pq^4 + \frac{5}{4}p^2q^4 + \frac{5}{12}q^6, \]
\[ R^{(3)}_1 = 1 + 2p + 2p^2 + \frac{4}{3}p^3 + \frac{23}{36}p^4 + \frac{2}{9}p^5 + \frac{1}{18}p^6 + \frac{1}{108}p^7 + \frac{1}{1296}p^8 + 2q^2 + 4pq^2 + \frac{23}{6}p^2q^2 + \frac{20}{9}p^3q^2 + \frac{5}{6}p^4q^2 + \frac{7}{36}p^5q^2 + \frac{7}{324}p^6q^2 + \frac{23}{12}q^4 + \frac{10}{3}pq^4 + \frac{5}{2}p^2q^4 + \frac{35}{36}p^3q^4 + \frac{35}{216}p^4q^4 + \frac{35}{108}p^5q^4 + \frac{35}{432}q^8 + \frac{5}{6}q^6 + \frac{35}{36}pq^6 + \frac{35}{108}p^2q^6 + \frac{35}{432}q^8. \]

Here denote \( R^{(0)}_1 \) as the MS-stability function of this method without any correction, namely the explicit Euler method, given by
\[ R^{(0)}_1 = (1 + p)^2 + q^2. \]
Figure 1 gives the MS-stable regions of the two-stage stochastic RK-type PC method (3.2) based on IRK2 (3.5). The MS-stable regions are the areas under the plotted lines and are symmetric about the $p$-axis. The MS-stability properties of this method with two corrections are better than those with one correction. The MS-stability properties of this method are much improved when the third correction is performed.

6. Numerical results. Numerical results for solving SDEs driven by one Wiener process are reported in this section. Denoting $y_N^{(i)}$ as the numerical approximation to $y^{(i)}(t_N)$ at step point $t_N$ in the $i$th simulation of all $K$ simulations, we use means of MS errors $MS$, strong order 1 rate $R_1$ and strong order 1.5 rate $R_{1.5}$, defined by

$$MS = \sqrt{\frac{1}{K} \sum_{i=1}^{K} (y_N^{(i)} - y^{(i)}(t_N))^2}, \quad R_1 = \frac{MS}{h}, \quad R_{1.5} = \frac{MS}{h^{1.5}},$$

to measure the accuracy and the convergence properties of the stochastic RK-type PC methods. All of the data in this section are based on 1000 simulated trajectories.

The first test equation is a nonlinear problem, whose Stratonovich form is

$$dy = -\alpha(1 - y^2)dt + \beta(1 - y^2) \circ dW(t), \quad y(0) = 0.5, \quad t \in [0, 1],$$

with $\alpha = -1$ and $\beta = 1$. The exact solution of this equation is [17]

$$y(t) = \frac{(1 + y_0)\exp(-2\alpha t + 2\beta W(t)) + y_0 - 1}{(1 + y_0)\exp(-2\alpha t + 2\beta W(t)) - y_0 + 1}.$$

Figure 2 gives the MS errors of the two stochastic RK-type PC methods based on IRK2 and IRK4, respectively, for solving the first test equation. For the two-stage PC method based on IRK2, the implicit corrector (3.5) is applied with a different number of corrections $l = 0, 1, 2, 3, 4$. From the left figure in Figure 2, the numerical
solution when no correction is performed, denoted as $l = 0$, does not converge to the exact solution of the corresponding Stratonovich SDE. The strong convergence rates of this method with $l = 1, 2, 3, \text{ or } 4$ are all equal to 1.0, as predicted by our theory. The averaged errors are smaller if more corrections are performed. The difference between the averaged errors of this method with three corrections and those with four corrections is small.

For the four-stage PC method based on IRK4, the implicit corrector is applied with a different number of corrections $l = 1, 2, 3$. From the right figure of Figure 2, the strong convergence rates of this method with $l = 1$ is equal to 1.0. When two corrections are performed, the strong convergence rate is between 1.0 and 1.5. The strong convergence rate of this method is 1.5 if three corrections are performed, which is again consistent with our theory.

It should be noticed that the accuracy of the stochastic RK-type PC method based on IRK4 with strong order 1.5 is not as good as that of the method based on IRK2 with strong order 1.0 when $2^{-10} \leq h \leq 2^{-6}$. The reason for this phenomenon is due to the eigenvalues of the method matrices. For IRK2, the eigenvalues of matrices $A$ and $B$ are

$$
\lambda(A) = \lambda(B) = \frac{1}{3} \pm \frac{\sqrt{3}}{6} i,
$$

while for IRK4 the eigenvalues of the method matrices are

$$
\lambda(A) = 0.400 \pm 0.622i, \quad -0.072 \pm 0.253i,
\lambda(B_1) = 0.096, \quad 0.333, \quad 0.878, \quad -0.0053,
\lambda(B_2) = 11.528, \quad -0.335, \quad -0.620 \pm 0.035i.
$$

The large eigenvalue of matrix $B_2$ causes amplifications in the errors of the PC method based on IRK4. This effect was well known in the deterministic case; see the work of Sommeijer [25].

In order to test out this supposition, we construct two methods, MIRK2 and MIRK4, which have strong order 1 and 1.5, respectively, and whose defining matrices

---

**Fig. 2.** MS errors for solving the first test equation.
have smaller spectral radius. The MIRK2 method is given by

\[
\begin{pmatrix}
-67 & 1000 & 1000 & -67 \\
1067 & -67 & 1000 & 1000 \\
3000 & 1000 & 3000 & 1000 \\
-933 & 1000 & -933 & 1000
\end{pmatrix}.
\]

The eigenvalues of the defined method matrices are

\[\lambda(A) = \lambda(B) = \frac{317 \pm \sqrt{11}}{1500}.\]

The MIRK4 method is different from the IRK4 method just in the method matrices \(B^{(1)}_M\) and \(B^{(2)}_M\), given by

\[
B^{(1)}_M = \begin{pmatrix}
-0.4103843710 & 0.2113248635 & 0.2566537645 & 0.1537306083 \\
1.1990595100 & 0 & -0.30000000 & -0.1103843746 \\
0.2807539857 & 0.4849084469 & 0.394375673 & -0.16000000 \\
0.2807539819 & 0.4849084506 & 0.394375673 & -0.16000000
\end{pmatrix},
\]

\[
B^{(2)}_M = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

whose eigenvalues are

\[\lambda(B^{(1)}_M) = -0.36 \pm 0.038i, 0.55, 0, \quad \lambda(B^{(2)}_M) = -0.76, 0, 0, 0.\]

Using the PC methods based on the correctors MIRK2 and MIRK4, we then repeated the calculations for solving the first test equation, and the numerical results are given in Figure 3. It is clear that the MIRK2 method is more effective than the IRK2 method, whose computational results are given by Figure 2. For four-stage correctors, the accuracy of the numerical results of MIRK4 is better than that of IRK4 with stepsize \(h = 2^{-6}, 2^{-7}, 2^{-8}\). When \(h = 2^{-9}\) and \(2^{-10}\), the accuracy of MIRK4 is just slightly better than that of IRK4.

The second test equation is also a nonlinear SDE, given by

\[dy = a(1 + y^2) \circ dt, \quad y(0) = 1, \quad t \in [0, 1],\]

with \(a = 0.1\). The exact solution is given in [17], namely

\[y = \tan(aW(t) + \arctan y_0).\]

Figure 4 gives the MS errors of the four PC methods for the second test equation. In this case the implicit corrector is applied with a different number of corrections \(l = 2, 3\). It is clear that the MIRK2 and MIRK4 methods with three corrections are much more effective than the IRK2 and IRK4 methods for the second test equation.

In order to discuss the relationship between the accuracy of the numerical methods and the computational cost, we use the following explicit two-stage RK methods to solve the first test equation:

\[
\begin{pmatrix}
-67 & 1000 & 1000 & -67 \\
1067 & -67 & 1000 & 1000 \\
3000 & 1000 & 3000 & 1000 \\
-933 & 1000 & -933 & 1000
\end{pmatrix},
\]
(1) The Heun method [12].

\begin{align}
Y &= y_n + hf(y_n) + \Delta W_n g(y_n), \\
y_{n+1} &= y_n + \frac{1}{2} h \left( f(y_n) + f(Y) \right) + \frac{1}{2} \Delta W_n \left( g(y_n) + g(Y) \right).
\end{align}

(2) The Burrages scheme [12].

\begin{align}
Y &= y_n + \frac{2}{3} hf(y_n) + \frac{2}{3} \Delta W_n g(y_n), \\
y_{n+1} &= y_n + h \left( \frac{1}{4} f(y_n) + \frac{3}{4} f(Y) \right) + \Delta W_n \left( \frac{1}{4} g(y_n) + \frac{3}{4} g(Y) \right).
\end{align}
Table 1 gives the accuracy and the computational cost, in terms of flops obtained by Matlab, of these explicit RK methods and those of the IRK2 and MIRK2 methods with three corrections. Clearly, both the IRK2 and MIRK2 methods with three corrections can achieve better accuracy than the other explicit methods with substantially reduced computation costs.

### Table 1: Accuracy and computational cost of some stochastic RK methods.

<table>
<thead>
<tr>
<th>Method</th>
<th>Heun</th>
<th>Burrages</th>
<th>Method 1</th>
<th>IRK2 ($l = 3$)</th>
<th>MIRK2 ($l = 3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Accuracy</td>
<td>4.9E-3</td>
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<td>4.7E-3</td>
<td>3.3E-3</td>
<td>5.6E-3</td>
</tr>
<tr>
<td>cost-flops</td>
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<td>7.9E+6</td>
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<tr>
<td>Accuracy</td>
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<td>4.3E-4</td>
<td>5.2E-4</td>
<td>3.6E-4</td>
<td>2.2E-4</td>
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<tr>
<td>cost-flops</td>
<td>5.7E+7</td>
<td>6.4E+7</td>
<td>3.3E+7</td>
<td>2.4E+7</td>
<td>2.4E+7</td>
</tr>
</tbody>
</table>

(3) Method 1 in [26].

\[
Y = y_n + \frac{3}{10} hf(y_n) + \frac{58}{100} \Delta W_n g(y_n),
\]

(6.4)

\[
y_{n+1} = y_n + h \left( \frac{56}{100} f(y_n) + \frac{44}{100} f(Y) \right) + \Delta W_n \left( \frac{4}{29} g(y_n) + \frac{25}{29} g(Y) \right).
\]

Table 1 gives the accuracy and the computational cost, in terms of flops obtained by Matlab, of these explicit RK methods and those of the IRK2 and MIRK2 methods with three corrections. Clearly, both the IRK2 and MIRK2 methods with three corrections can achieve better accuracy than the other explicit methods with substantially reduced computation costs.

The third test equation is given by

\[
dy_1 = y_2 dt + \theta y_2 \circ dW(t),
\]

(6.5)

\[
dy_2 = \mu \left( (1 - y_1^2) y_2 - y_1 \right) + \theta \left( (1 - y_1^2) y_2 - y_1 \right) \circ dW(t).
\]

This equation is the ordinary Van der Pol equation [14] when $\theta = 0$. The Van der Pol equation is stiff when $\mu$ is large.

We use IRK2 with $l = 3$ to solve this equation. In Figure 5 we give four simulations of this equation. The top two simulations in Figure 5 are obtained with parameters $\mu = 1$, $\theta = 0.1$, and $\theta = 1$ and stepsize $h = 0.01$. The bottom two simulations are obtained with parameters $\mu = 10$, $\theta = 0.1$, and $\theta = 0.5$ and stepsize $h = 0.001$. The numerical simulations with $\theta = 0.1$ are similar to those of the deterministic Van der Pol equation with the same $\mu$.

In order to discuss the efficiency of the two-stage PC methods, we use IRK2 ($l = 3$) with stepsize $h = 0.0001$ to get a numerical solution which is regarded as the “accurate” solution in the case of $\mu = 1$ and different $\theta$. We compare this “accurate” solution with the numerical simulations obtained by the explicit RK methods (6.2), (6.3), and (6.4) and those obtained by IRK2 ($l = 3$) and MIRK2 ($l = 3$). Numerical results presented in Figure 6 are based on 100 simulations. The left figure of Figure 6 gives the accuracy of numerical solutions with $\mu = 1$ and $0.1 \leq \theta \leq 1.0$. The accuracy of numerical simulations of IRK2 ($l = 3$) and MIRK2 ($l = 3$) is considerably better than those of the other methods. In the right figure of Figure 6, we present the proportions of “acceptable” solutions with the standard that the averaged error is less than 1.0. It should be noticed that the proportions are dependent on the standard.

We can get more “acceptable” simulations by IRK2 ($l = 3$) and MIRK2 ($l = 3$) than with the other explicit RK methods. The explicit RK methods are not suitable for solving this equation with values for $\theta > 1$.

Similar numerical results about the accuracy and the proportions of acceptable solutions can be also obtained for the case $\mu = 10$ and $\theta \in [0.1, 1.0]$. In this case a smaller stepsize, for example $h = 0.00001$, should be used for the “accurate solution.”
7. Conclusions. In this paper we have constructed PC methods based on the trivial predictor and stochastic implicit RK correctors for solving SDEs. Using the colored rooted tree theory and stochastic B-series, we present an order condition theorem for constructing stochastic RK-type PC methods. We also present detailed order conditions of the stochastic RK-type PC methods with strong convergence order 1.0 and 1.5. Two two-stage implicit RK methods with strong global order 1.0 and two four-stage implicit RK methods with strong global order 1.5 are constructed in this paper. The following conclusions can be made from the stability analysis and numerical behavior of the RK-type PC methods presented in this paper.

(1) As the number of parameters is larger than the number of order conditions, additional conditions can be used to determine the coefficients of stochastic RK methods in order to get better stability properties and numerical behavior. For example, we may consider a two-stage implicit RK method which has good stability properties at infinity. Applying this method (3.1) to the linear test equation (5.1) gives

\[ y_{n+1} = R(p, q)y_n, \]
where

\[ R(p, q) = 1 + (p\alpha^\top + qJ_1\beta^\top)(I - pA - qJ_1B)^{-1}e. \]

This method will have damping stability properties at infinity if

\[ \alpha^\top A^{-1}e = 1, \quad \beta^\top B^{-1}e = 1. \]

The implicit two-stage RK method (3.5) satisfies these conditions.

(2) Another possible way to improve the stability properties and the numerical behavior of stochastic RK-type PC methods is to reduce the magnitude of the eigenvalues of the matrices in the stochastic RK methods. The cue is in the expression of \( R^{(1)} (5.2) \). In order to verify this supposition, we construct two methods, MIRK2 and MIRK4. Compared with IRK2 and IRK4, the eigenvalues of the method matrices in MIRK2 and MIRK4 are small in magnitude. Numerical results of the MIRK2 and MIRK4 methods are more accurate than those of IRK2 and IRK4. The effect has also been observed in the deterministic case.

(3) The stochastic RK-type PC methods are more effective than other explicit stochastic RK methods. For two-stage RK methods with strong order 1.0, the superiority of the PC method based on IRK2 or MIRK2 is due to the better stability properties (shown in Figures 1 and 6), the better accuracy, and the less computational cost (shown in Table 1 and Figure 6). For the RK methods with strong order 1.5, the PC method will be more effective than the explicit RK methods with the same order if it is implemented on a parallel computer.

Thus we may consider stochastic RK-type PC methods which have better stability properties and numerical behavior by adding additional conditions or by reducing the magnitude of the eigenvalues of the matrices in the stochastic RK methods. In addition, we can apply splitting techniques [20] to implicit RK methods to construct numerical schemes which are suitable for solving stiff SDEs. Finally, we note that these concepts can be applied to SDE problems driven by more than one Wiener process. However, spatial constraints for this work means that all of these are topics for future work.
REFERENCES