CALCULATING CURRENT DENSITIES AND FIELDS PRODUCED
BY SHIELDED MAGNETIC RESONANCE IMAGING PROBES∗

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Abstract. A method is presented for computing the fields produced by radio frequency probes
of the type used in magnetic resonance imaging. The effects of surrounding the probe with a shielding
coil, intended to eliminate stray fields produced outside the probe, are included. An essential feature
of these devices is the fact that the conducting rungs of the probe are of finite width relative to the
coil radius, and it is therefore necessary to find the distribution of current within the conductors
as part of the solution process. This is done here using a numerical method based on the inverse
finite Hilbert transform, applied iteratively to the entire structure including its shielding coils. It
is observed that the fields are influenced substantially by the width of the conducting rungs of the
probe, since induced eddy currents within the rungs become more pronounced as their width is
increased. The shield is also shown to have a significant effect on both the primary current density
and the resultant fields. Quality factors are computed for these probes and compared with values
measured experimentally.

Key words. radio-frequency probe, quality factor, integral equations, inverse finite Hilbert
transform

AMS subject classifications. 78A55, 78A50, 65R20, 45F15

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1. Introduction. Nuclear magnetic resonance (NMR) is now an established
technique for spatial imaging, and its use in medicine, in particular, is becoming
more widespread. Nonmedical uses of the technology are likewise becoming more
prevalent, and a recent review article by Gladden [11] is devoted to the applica-
tions of magnetic resonance imaging (MRI) in chemical engineering, for example.
The success of the technology is related to its ability to create accurate spatial im-
ages.

The principle governing relationship in NMR is the Lamor equation,
\[ \omega = \gamma B_0, \]
in which \( \omega \) is the Lamor precessional frequency, \( \gamma \) is the nuclei-specific gyromagnetic
ratio, and \( B_0 \) is the applied magnetic field. This equation refers to the situation where
an ensemble of nuclei possessing nuclear spin are subjected to a strong magnetic field.
A number of possible energy levels are developed by the interaction of the applied
field and the nuclear spins, since these possess magnetic moments. In order to induce
transitions between these energy levels, radio frequency (RF) energy is applied to the
ensemble orthogonal to the direction of the applied field at the Lamor precessional
frequency. After the RF excitation ceases, the spin ensemble tends to return to its
original state and in doing so emits energy; this is the NMR signal. This signal can be
detected by the same device (termed an RF probe) that was used to transmit the RF
excitation, or by a separate probe. In either case the probe(s) are tuned to operate
at or near the Lamor frequency.
Spatial encoding of the NMR signal is achieved with the use of magnetic field gradients. These gradients are designed to provide a linear variation of the longitudinal magnetic field ($B_z$) across the sample. According to the Lamor equation, the precessional frequency is now spatially encoding, and spatial information is revealed by multidimensional Fourier transformation, resulting in a spin density map of the sample.

It is most important in NMR and MRI imaging experiments to attempt to maximize the signal-to-noise ratio (SNR) of the experiment and to irradiate all parts of the sample with the same-strength RF field. Similarly, it is important that the NMR signal from all parts of the sample be received with the correct weighting by the RF probe. Perhaps the two most important characteristics of an RF probe are then the provision of a homogeneous field in the volume of the coil and the efficient generation of a strong RF field, implying a high quality factor $Q$. By reciprocity, if a coil provides homogeneous excitation, it will also receive NMR signals in a homogeneous fashion [15]. Therefore, it will be assumed here that descriptions of excitation distributions of coils apply with equal relevance to their use as NMR receivers.

A popular form of the RF coil, or resonator, consists of a number of conducting rungs arranged around the circumference of a circular cylinder and running parallel to the axis of the cylinder. These structures have a certain inductance. In order to generate an RF field that is homogeneous in any transverse plane (one that is orthogonal to the axis of the coil), it is first necessary to establish a sinusoidal current distribution around the circumference of the cylinder at one of its ends, and this sinusoidal pattern of current is then propagated along the cylinder by means of the longitudinal conducting rungs. Typically, this current distribution is established by forming an RF standing wave around the periphery of the coil at one end, using appropriate capacitors to separate each of the longitudinal conducting rungs; in this way, a tuned resonator is created, since both inductive and capacitive elements are present. Alternatively, capacitors could be included in the longitudinal elements, which would then be connected at their ends. Such arrangements are commonly known as bird-cage resonators (see Hayes et al. [14]).

In much of the early work in RF probe design for NMR/MRI (see [1], [4], [12], [14]), the fields generated within the probe were calculated simply using the magnetostatic Biot–Savart law, under the assumption that the current is uniformly distributed across each of the conducting rungs. This work has recently been extended by Mahony et al. [16] to cope with the case when RF current flows in a bird-cage coil. In this work, it was assumed that the conducting rungs of the probe could be regarded simply as wires, and the resulting field was computed using a somewhat novel retarded potential approach. Nevertheless, in the practical construction of resonators for medium-to high-frequency spectroscopy and imaging, the requirement for very low inductance designs means that streamline (or large width, flat sheet) conductors are preferable. (Modern NMR technology operates up to frequencies of 750 MHz.) Under such circumstances, it is known that the current does not flow uniformly in the azimuthal direction, and its density can approach very large values at the edges of the conducting strips, as indicated by Carlson [6], for example.

It is therefore a major aim of the present study to develop methods for computing the current density in each of the longitudinal conductors when these are of finite width relative to the radius of the cylindrical probe. The nett (integrated) current in each may be assumed to be known by virtue of the known characteristics of the RF standing wave of current produced at one end of the probe. From a mathematical
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point of view, it is therefore necessary to solve a classical, ill-posed problem of inverse type, in which current densities are to be reconstructed from knowledge of the total currents. This is expressed as a system of first-kind Fredholm integral equations, which are notoriously difficult to solve (see Delves and Mohamed [9]). Nevertheless, considerable success has been achieved recently by Forbes, Crozier, and Doddrell [10] in a study of unshielded probes, and the importance of accounting for the uneven current distribution in the conducting rungs of the probe, when computing magnetic and electric fields, was demonstrated. A comprehensive review of earlier work involving analytical solutions for diffraction around a single strip is given by Bowman, Senior, and Uslenghi [5].

An RF probe is generally provided with RF shields that are external to the probe and lie around the circumference of a larger cylinder co-axial with the probe. These shields provide an RF ground plane for the resonant cavity, and prevent flux from escaping into the equipment surrounding the probe, particularly the gradient coils, shim sets, and the magnet itself. By reciprocity, shields prevent external signals from being received by the probe. An effective shield should maintain the quality factor $Q$ of the resonator, prevent spurious resonances, and reduce the detuning of the structure as it is placed intimately with surrounding structures; this intimate positioning is necessary to ensure maximum access to the magnet of the NMR device. The effects of shields on the current distributions in the primary probe and on the homogeneity of the fields generated within the probe are largely unknown to date and will be studied in this paper.

The pulsed magnetic gradient fields that are used in conjunction with the RF probe for imaging experiments can generate significant eddy currents within the shields, and this is often overcome in practice by cutting narrow slots into the shields [13]. A photograph of such a shielding arrangement is presented in Figure 1. Here, it is possible to see the longitudinal rungs on the inner primary probe, in addition to the outer shielding coils, in which several fine slots have been inscribed. Although there is often no nett current flowing in the shields, it is nevertheless the case that local eddy currents can be induced in the shields by the RF signal in the rungs of the probe, and this effect must be included in an accurate model of shielded RF probes if reliable fields are to be computed within these devices.

It is therefore the purpose of this paper to develop reliable methods for calculating the current densities in the rungs of the probe and current densities induced in the shields, in addition to determining accurate fields within the probe itself. In particular, the influence of the shields upon the properties of the resonator will be investigated. Methods for determining the quality factor $Q$ for the coil will be demonstrated, and the results compared with $Q$-factors actually measured in devices of this type.

2. The mathematical model. We consider a circularly cylindrical probe in which the primary conducting rungs are placed at radius $a$ from the central axis of the device, as illustrated in Figure 2. There are $N$ rungs, labeled 1, 2, ..., $N$ counterclockwise around the $z$-axis, as shown, so that a cross section through the device lies in the $x$-$y$ plane. In general, there will also be $L$ shielding segments present at radius $b$ from the $z$-axis, and these are likewise labeled from 1 to $L$ in a counterclockwise direction.

It will be assumed for simplicity in the present analysis that field and current variations are purely sinusoidal in the $z$-direction (normal to the plane of Figure 2)
FIG. 1. Photograph of shielded “bird-cage”–type RF resonators, showing the shields and the rungs on the interior primary coil.

FIG. 2. Sketch of the cross section of a resonator, showing the rungs on the primary and shielding coils.
so that the transverse electric and magnetic (TEM) approximation can be invoked. It follows that the electric field $E$, the magnetic induction field $B$, and the current density $j$ per unit cross-sectional width of the conducting strip can be expressed in the (approximate) forms

\[ E(x, y, z, t) = E_T(x, y) \exp(i\omega[\sqrt{\mu/\epsilon}z - t]), \]
\[ B(x, y, z, t) = B_T(x, y) \exp(i\omega[\sqrt{\mu/\epsilon}z - t]), \]
\[ j(x, y, z, t) = j_T(x, y) \exp(i\omega[\sqrt{\mu/\epsilon}z - t]), \]

where the magnetic permeability and electric permittivity of the air surrounding the conducting strips are $\mu$ and $\epsilon$, respectively, the frequency of the signal is $\omega$, and $t$ denotes time. In this TEM approximation, the transverse parts $E_T$ and $B_T$ of the electric and magnetic fields have no axial component, so

\[ E_T \cdot k = 0 \quad \text{and} \quad B_T \cdot k = 0, \]

and $k$ denotes the unit vector pointing in the $z$-direction.

The TEM approximation would not, in fact, be valid within a completely closed conducting cylinder, since it would fail to satisfy the required boundary conditions (see, for example, Balanis [2]). However, in view of the gaps between the rungs and between the shielding segments, a shielded RF probe is (of course) not closed; this is enhanced by the fact that different currents flow in neighboring rungs. For this reason, the TEM assumption is expected to provide a reasonable approximation in this application. In addition, the current distribution on the rungs is often such as to excite the lowest mode within the coil, for which at least the magnetic field is well described by the TEM approximation, and this is confirmed by experiment [8].

A consequence of the assumption (1) of TEM-mode solutions is that the full system of governing equations (Maxwell’s equations) possesses solutions in which there is a simple relationship between the magnetic and electric fields, given by

\[ B_T = -\sqrt{\mu/\epsilon}(k \times E_T) \]

(see, e.g., Ramo, Whinnery, and Van Duzer [18]). Furthermore, it follows from equations (1) and (2) that Faraday’s law reduces to

\[ \nabla_2 \times E_T = 0, \]

from which a scalar potential $\Phi$ can be defined immediately for the electric field, according to the relation $E_T = -\nabla_2 \Phi$. Here, $\nabla_2 = (\partial/\partial x, \partial/\partial y)$ is the gradient operator in the transverse plane shown in Figure 2.

For the purposes of computing the electric and magnetic fields within the probe and the current densities within the longitudinal rungs, it will be assumed that the copper rungs in the resonator are perfect conductors. The boundary condition to be imposed is therefore that $\Phi$ must be constant along the surface of each conductor. Once the scalar potential $\Phi$ has been determined, the transverse part $j_T$ of the current density at the surface of the conductor may be determined according to

\[ j_T = \sqrt{\mu/\epsilon}(n \cdot \nabla_2 \Phi)k, \]

where $n$ represents the normal to the conducting surface.

Since the interior of the MRI probe does not possess sources of charge, Maxwell’s equations also reveal that the transverse part of the electric field is solenoidal, so
\( \nabla^2 \mathbf{E} = 0 \) under the TEM approximation (1) and the relationship (2). It follows that the scalar potential \( \Phi \) satisfies Laplace’s equation \( \nabla^2 \Phi = 0 \) in the two-dimensional region illustrated in Figure 2. Thus the potential \( \Phi \) may be regarded as the real part of an analytic complex function of coordinate \( x + iy \), and this fact was exploited by Forbes, Crozier, and Doddrell [10] in the case of unshielded coils, where conformal mapping techniques were used to calculate the current densities within the primary rungs of the resonator.

In this study, the presence of the shielding coils at radius \( r = b \) reduces the usefulness of conformal mapping methods rather severely, for reasons that need not be discussed further here, and so a direct solution method in the unmapped \( x-y \) plane is used instead. The transverse component of the current density per unit width on the primary rungs (at radius \( r = a \)) is written as \( \mathbf{j}_T = j_{t,n}^P \mathbf{k} \), \( n = 1, 2, \ldots, N \), and the current density on the shielding rungs (at \( r = b \)) is \( \mathbf{j}_T = j_{t,l}^S \mathbf{k} \), \( l = 1, 2, \ldots, L \); since the scalar potential \( \Phi \) satisfies Laplace’s equation, it is possible at once to write down a relationship between the potential and the current densities in the form

\[
\Phi(x, y) = \frac{1}{\pi} \sqrt{\frac{\mu}{\varepsilon}} \sum_{n=1}^{N} \int_{s_{n,1}}^{s_{n,2}} j_{t,n}^P(s_n) \ln \frac{1}{a} \sqrt{(x-x(s_n))^2 + (y-y(s_n))^2} \, ds_n
\]

\[
+ \frac{1}{\pi} \sqrt{\frac{\mu}{\varepsilon}} \sum_{l=1}^{L} \int_{s_{l,1}}^{s_{l,2}} j_{t,l}^S(s_l) \ln \frac{1}{b} \sqrt{(x-x(s_l))^2 + (y-y(s_l))^2} \, ds_l.
\]

This result has been derived using the boundary condition (3). In this equation, the symbols \( s_n \) and \( s_l \) refer to arclengths around the \( n \)th primary rung and the \( l \)th shielding rung, respectively, and the beginning and end points of these rungs are represented by \( s_{n,1} \) and \( s_{n,2} \) for the \( n \)th primary rung and \( s_{l,1} \) and \( s_{l,2} \) for the \( l \)th shielding rung.

In view of the fact that the MRI resonator involves circular geometry, it is appropriate to convert to cylindrical polar coordinates \((r, \theta)\), in terms of which the coordinates of the field point are \( x = r \cos \theta \) and \( y = r \sin \theta \), as usual. The beginning and end points of the \( n \)th primary rung are \( s_{n,2} = a \theta_{n,2} \) and \( s_{n,1} = a \theta_{n,1} \) (since the rungs are traversed in a counterclockwise direction, as in Figure 2), and the corresponding points for the \( l \)th shielding rung are \( s_{l,2} = b \theta_{l,2} \) and \( s_{l,1} = b \theta_{l,1} \). The length along the \( n \)th primary rung is \( s_n = a \beta_n \), and along the \( l \)th shielding rung it is \( s_l = b \psi_l \). In polar coordinates, the solution (4) becomes

\[
\Phi(r, \theta) = -\frac{a}{\pi} \sqrt{\frac{\mu}{\varepsilon}} \sum_{n=1}^{N} \int_{\theta_{n,1}}^{\theta_{n,2}} j_{t,n}^P(\beta_n) \ln \frac{1}{a} \sqrt{r^2 + a^2 - 2ra \cos(\beta_n - \theta)} \, d\beta_n
\]

\[
- \frac{b}{\pi} \sqrt{\frac{\mu}{\varepsilon}} \sum_{l=1}^{L} \int_{\theta_{l,1}}^{\theta_{l,2}} j_{t,l}^S(\psi_l) \ln \frac{1}{b} \sqrt{r^2 + b^2 - 2rb \cos(\psi_l - \theta)} \, d\psi_l.
\]

A system of integral equations for the current densities \( j_{t,n}^P \) and \( j_{t,l}^S \) in the primary and shielding coils, respectively, may be derived from the system (5) by applying the boundary conditions

\[
\lim_{r \to a} \frac{\partial \Phi}{\partial \theta_m} = 0 \quad \text{in} \quad \theta_{m,1} < \theta_m < \theta_{m,2}, \quad m = 1, 2, \ldots, N,
\]
on the (inner) primary rungs and

\[
\lim_{r \to b} \frac{\partial \Phi}{\partial \theta_k} = 0 \quad \text{in } \theta_{k,1} < \theta_k < \theta_{k,2}, \quad k = 1, 2, \ldots, L, \tag{6b}
\]

for each of the shielding rungs. After some algebra, a coupled system of \(N + L\) equations is obtained, which it is convenient to write in the form

\[
\text{CPV} \int_{\theta_{m,1}}^{\theta_{m,2}} j_{t,m}^P(\beta) \cot \left( \frac{\beta - \theta_m}{2} \right) d\beta = F_m(\theta_m), \quad m = 1, 2, \ldots, N, \tag{7}
\]

\[
\text{CPV} \int_{\theta_{k,1}}^{\theta_{k,2}} j_{t,k}^S(\psi) \cot \left( \frac{\psi - \theta_k}{2} \right) d\psi = G_k(\theta_k), \quad k = 1, 2, \ldots, L,
\]

where the functions appearing on the right-hand side of these equations have been defined to be

\[
F_m(\theta_m) = -\sum_{\substack{n \neq m \leq N \atop (n \neq m)}} \int_{\theta_{n,1}}^{\theta_{n,2}} j_{t,n}^P(\beta) \cot \left( \frac{\beta - \theta_m}{2} \right) d\beta
\]

\[
- \frac{2b}{a} \sum_{l=1}^{L} \int_{\theta_{l,1}}^{\theta_{l,2}} j_{t,l}^S(\psi) \frac{ab \sin(\psi - \theta_m)}{a^2 + b^2 - 2ab \cos(\psi - \theta_m)} d\psi
\]

and

\[
G_k(\theta_k) = -\sum_{\substack{l \neq k \leq L \atop (l \neq k)}} \int_{\theta_{l,1}}^{\theta_{l,2}} j_{t,l}^S(\psi) \cot \left( \frac{\psi - \theta_k}{2} \right) d\psi
\]

\[
- \frac{2a}{b} \sum_{n=1}^{N} \int_{\theta_{n,1}}^{\theta_{n,2}} j_{t,n}^P(\beta) \frac{ba \sin(\beta - \theta_k)}{b^2 + a^2 - 2ba \cos(\beta - \theta_k)} d\beta.
\]

In the system (7) of governing equations, the integrals that are singular in the Cauchy principal value (CPV) sense have been written on the left side, since these are deserving of special attention. Nevertheless, the system (7) is genuinely a coupled one, since all the unknown current densities appear in each equation. When written in the form (7), it is evident that the system essentially consists of Fredholm integral equations of the first kind for each of the unknown current densities, and such equations are generally extremely difficult to solve, because of their ill-conditioned nature (see Delves and Mohamed [9]). A direct finite-difference solution of Laplace’s equation for the scalar potential \(\Phi\) has been undertaken by Carlson [6], for example, and is equivalent to a straightforward numerical solution of the system (7); it led to current densities possessing spurious oscillations. Similar results were obtained in Forbes, Crozier, and Doddrell [10] with an iterative solution of a system of integral equations similar to (7) and is evidence of the difficulties posed by the ill-conditioning of the system.

In a second numerical scheme, Forbes, Crozier, and Doddrell [10] demonstrated that the iterative application of the inverse finite Hilbert transform to each of the equations in a system similar to (7) gave superior results in which the characteristic numerical oscillations were eliminated entirely. Clearly, it would be desirable to apply
these methods here also, but the difficulty is that the integral equations (7) are not of a form for which the inverse Hilbert transform can be applied immediately. This can be overcome, however, simply by rewriting the system as

\[ \int_{\theta_m,1}^{\theta_m,2} j_{l,m}^P(\beta) \frac{2}{\beta - \theta_m} d\beta = F_m(\theta_m) - \int_{\theta_m,1}^{\theta_m,2} j_{l,m}^P(\beta) \left[ \cot \left( \frac{\beta - \theta_m}{2} \right) - \frac{2}{\beta - \theta_m} \right] d\beta, \quad m = 1, 2, \ldots, N, \]

\[ \int_{\theta_k,1}^{\theta_k,2} j_{l,k}^S(\psi) \frac{2}{\psi - \theta_k} d\psi = G_k(\theta_k) - \int_{\theta_k,1}^{\theta_k,2} j_{l,k}^S(\psi) \left[ \cot \left( \frac{\psi - \theta_k}{2} \right) - \frac{2}{\psi - \theta_k} \right] d\psi, \quad k = 1, 2, \ldots, L. \]

The formal solution to the equations in the system (9) is given in terms of an inverse finite Hilbert transform and may be found in Carrier, Krook, and Pearson [7, p. 424] for example. This then yields

\[ j_{l,m}^P(\theta_m) = -\frac{1}{2\pi^2} \frac{\sqrt{\theta_m,2 - \theta_m}}{\theta_m - \theta_m,1} \int_{\theta_m,1}^{\theta_m,2} \frac{d\beta}{\beta - \theta_m} \sqrt{\beta - \theta_m,1 \over \beta - \theta_m} \times \left\{ F_m(\beta) - \int_{\theta_m,1}^{\theta_m,2} j_{l,m}^P(\alpha) \left[ \cot \left( \frac{\alpha - \beta}{2} \right) - \frac{2}{\alpha - \beta} \right] d\alpha \right\} + \frac{K_m}{\sqrt{\theta_m,2 - \theta_m,1}}, \quad m = 1, 2, \ldots, N, \]

and

\[ j_{l,k}^S(\theta_k) = -\frac{1}{2\pi^2} \frac{\sqrt{\theta_k,2 - \theta_k}}{\theta_k - \theta_k,1} \int_{\theta_k,1}^{\theta_k,2} \frac{d\psi}{\psi - \theta_k} \sqrt{\psi - \theta_k,1 \over \psi - \theta_k} \times \left\{ G_k(\psi) - \int_{\theta_k,1}^{\theta_k,2} j_{l,k}^S(\tau) \left[ \cot \left( \frac{\tau - \psi}{2} \right) - \frac{2}{\tau - \psi} \right] d\tau \right\} + \frac{C_k}{\sqrt{\theta_k,2 - \theta_k,1}}, \quad k = 1, 2, \ldots, L. \]

In these expressions, the \( N + L \) constants \( K_m \) and \( C_k \) are arbitrary, since it is known that equations of the form (9) possess eigensolutions with square-root singularities at each end of the interval of integration; these are the second terms in each of equations (10a) and (10b), and they will be seen to play a crucial role in the results to follow. In fact, equations of the form (9) first arose in airfoil theory and hence are known as airfoil equations (see, e.g., Batchelor [3, p. 468]), and in that application, they give the relationship between slope and circulation on the wing. (The square-root singularities at the edges of the rungs are a special case of Meixner’s edge condition; see Mittra and Lee [17].)
In order to determine the constants $K_m$ and $C_k$ in (10), further $N + L$ conditions on the solution are required. This information is provided by the fact that the nett currents are known in advance on each rung in the primary and shielding coils. It follows that

\[ I_m^P = 2 \int_{\theta_{m,1}}^{\theta_{m,2}} j_{l,m}^P(\theta_m) \, d\theta_m, \quad m = 1, 2, \ldots, N, \]  

are the overall currents on the $N$ primary rungs, and

\[ I_k^S = 2 \int_{\theta_{k,1}}^{\theta_{k,2}} j_{l,k}^S(\theta_k) \, b \, d\theta_k, \quad k = 1, 2, \ldots, L, \]

are the currents imposed on the $L$ shielding segments. The factors of 2 appearing in (11a) and (11b) allow for the fact that current flows down both sides of the conducting rung.

The complete solution for the current densities $j_{l,m}^P$ in the primary and $j_{l,k}^S$ in the shielding coils is therefore given in implicit form by (10) and (11). In view of the fact that the unknown current densities appear on both sides of these equations, it is necessary to solve them using iteration, and in practice, a straightforward fixed-point method is sufficient.

There is an advantage to incorporating the forms of the right-hand side functions $F_m$ and $G_k$ from (8) into the solutions (10) and (11) so that certain singular integrals that arise in the solution may be treated explicitly. This involves considerable algebraic manipulation, and results in the integral equations (10) taking the final forms

\[ j_{l,m}^P(\theta_m) = \frac{1}{\pi^2} \sqrt{\frac{\theta_{m,2} - \theta_{m,1}}{\theta_{m,1} - \theta_{m,2}}} \left\{ \int_{\theta_{m,1}}^{\theta_{m,2}} j_{l,m}^P(\alpha)Y_1(R_3, S_3, T_3) \, d\alpha ight. 
\]  
\[ + \sum_{n \neq m}^{N} \int_{\theta_{n,1}}^{\theta_{n,2}} j_{l,n}^P(\alpha)Y_3(R_3, S_3, T_3) \, d\alpha 
\]  
\[ + \frac{2b}{a} \sum_{l=1}^{L} \int_{\theta_{l,1}}^{\theta_{l,2}} j_{l,k}^S(\tau)Y_4(R_4, S_3, T_3; a, b) \, d\tau \left. \right\} + \frac{K_m}{\sqrt{\theta_{m,2} - \theta_m} \theta_m - \theta_{m,1}}, \quad m = 1, 2, \ldots, N, \]  

and

\[ j_{l,k}^S(\theta_k) = \frac{1}{\pi^2} \sqrt{\frac{\theta_{k,2} - \theta_{k,1}}{\theta_{k,1} - \theta_{k,2}}} \left\{ \int_{\theta_{k,1}}^{\theta_{k,2}} j_{l,k}^S(\tau)Y_1(R_7, S_7, T_7) \, d\tau 
\]  
\[ + \sum_{l \neq k}^{L} \int_{\theta_{l,1}}^{\theta_{l,2}} j_{l,k}^S(\tau)Y_3(R_7, S_7, T_7) \, d\tau 
\]  
\[ + \frac{2a}{b} \sum_{n=1}^{N} \int_{\theta_{n,1}}^{\theta_{n,2}} j_{l,n}^P(\alpha)Y_4(R_8, S_7, T_7; b, a) \, d\alpha \left. \right\} + \frac{C_k}{\sqrt{\theta_{k,2} - \theta_k} \theta_k - \theta_{k,1}}, \quad k = 1, 2, \ldots, L. \]
where we have defined the functions

\[
\begin{align*}
R_3 &= \alpha - \frac{1}{2}(\theta_{m,1} + \theta_{m,2}), \\
S_3 &= \frac{1}{2}(\theta_{m,2} - \theta_{m,1}), \\
T_3 &= \left[ \frac{1}{2}(\theta_{m,1} + \theta_{m,2}) - \theta_m \right] / S_3, \\
R_4 &= \tau - \frac{1}{2}(\theta_{m,1} + \theta_{m,2})
\end{align*}
\]

and

\[
\begin{align*}
R_7 &= \tau - \frac{1}{2}(\theta_{k,1} + \theta_{k,2}), \\
S_7 &= \frac{1}{2}(\theta_{k,2} - \theta_{k,1}), \\
T_7 &= \left[ \frac{1}{2}(\theta_{k,1} + \theta_{k,2}) - \theta_k \right] / S_7, \\
R_8 &= \alpha - \frac{1}{2}(\theta_{k,1} + \theta_{k,2})
\end{align*}
\]

The singular integrals \( Y_1, Y_3, \) and \( Y_4 \) appearing as kernels in the integral equations (12) are defined to be

\[
\begin{align*}
Y_1(R_3, S_3, T_3) &= \frac{1}{2} \text{CPV} \int_{\theta_{m,1}}^{\theta_{m,2}} \sqrt{\beta - \theta_{m,1}} \left[ \cot \left( \frac{\alpha - \beta}{2} \right) - \frac{2}{\alpha - \beta} \right] \frac{d\beta}{\beta - \theta_m}, \\
Y_3(R_3, S_3, T_3) &= \frac{1}{2} \text{CPV} \int_{\theta_{m,1}}^{\theta_{m,2}} \sqrt{\beta - \theta_{m,1}} \cot \left( \frac{\alpha - \beta}{2} \right) \frac{d\beta}{\beta - \theta_m}, \\
Y_4(R_4, S_3, T_3; a, b) &= \frac{1}{2} \text{CPV} \int_{\theta_{m,1}}^{\theta_{m,2}} \sqrt{\beta - \theta_{m,1}} \frac{ab \sin(\tau - \beta)}{\alpha^2 + b^2 - 2ab \cos(\tau - \beta)} \frac{d\beta}{\beta - \theta_m},
\end{align*}
\]

and in the appendix, we show how to transform these into nonsingular integrals with periodic integrands that are in a form ideally suited to accurate evaluation simply using the trapezoidal rule.

The constants \( K_m \) and \( C_k \) in the integral equations (12) are likewise found to be

\[
\begin{align*}
K_m &= \frac{P_m}{2\pi a} - \frac{2S_3}{\pi^2} \left\{ \int_{\theta_{m,1}}^{\theta_{m,2}} j_{l,m}(\alpha) Y_2(R_3, S_3) \, d\alpha \\
&\quad + \sum_{n=1}^{N} \int_{\theta_{m,1}}^{\theta_{m,2}} j_{l,n}(\alpha) Y_5(R_3, S_3) \, d\alpha \\
&\quad + 2b \sum_{l=1}^{L} \int_{\theta_{m,1}}^{\theta_{m,2}} j_{l,1}(\tau) Y_6(R_4, S_3; a, b) \, d\tau \right\}, \\
&\quad m = 1, 2, \ldots, N,
\end{align*}
\]
and

\[ C_k = \frac{I_k}{2\pi b} \left\{ \int_{\theta_{k,1}}^{\theta_{k,2}} j_{l,k}^S(\tau) Y_2(R_\tau, S_\tau) d\tau \right\} \]

\[ + \sum_{l=1}^{L} \int_{\theta_{k,1}}^{\theta_{l,2}} j_{l,k}^S(\tau) Y_5(R_\tau, S_\tau) d\tau \]

\[ + \frac{2\alpha}{b} \sum_{n=1}^{N} \int_{\theta_{n,1}}^{\theta_{n,2}} j_{l,n}^P(\alpha) Y_6(R_\tau, S_\tau; b, a) d\alpha \}, \]

\( k = 1, 2, \ldots, L. \)

Here, the functions \( R_3 \) and so on are as given in (13), and the integrals \( Y_2, Y_5, \) and \( Y_6 \) are defined to be

\[ Y_2(R_3, S_3) = \frac{1}{4S_3} \int_{\theta_{m,1}}^{\theta_{m,2}} \sqrt{\frac{\beta - \theta_{m,1}}{\theta_{m,2} - \beta}} \left[ \cot \left( \frac{\alpha - \beta}{2} \right) - \frac{2}{\alpha - \beta} \right] d\beta, \]

\[ Y_5(R_3, S_3) = \frac{1}{4S_3} \int_{\theta_{m,1}}^{\theta_{m,2}} \sqrt{\frac{\beta - \theta_{m,1}}{\theta_{m,2} - \beta}} \cot \left( \frac{\alpha - \beta}{2} \right) d\beta, \]

and

\[ Y_6(R_4, S_3; a, b) = \frac{1}{4S_3} \int_{\theta_{m,1}}^{\theta_{m,2}} \sqrt{\frac{\beta - \theta_{m,1}}{\theta_{m,2} - \beta}} \frac{ab \sin(\tau - \beta)}{a^2 + b^2 - 2ab \cos(\tau - \beta)} d\beta. \]

As above, these integrals (18)–(20) may be written in a form that enables them to be evaluated easily and accurately using the trapezoidal rule, using the transformations outlined in the appendix.

The numerical solution of the system of integral equations (12) and supplementary conditions (17) is now reasonably straightforward. An initial guess is made for the current densities \( j_{l,m}^P \) and \( j_{l,k}^S \) at some number \( R \) of grid points on each rung. (Here, \( R = 101 \) is usually found to be sufficient.) The constants \( K_m \) and \( C_k \) are evaluated from (17), and the integral equations (12) then provide an improved estimate for the current densities. This whole process is repeated until the root mean square difference between estimates for the current densities has fallen below some preset tolerance level (usually \( 10^{-7} \)), at which point this fixed-point iteration scheme is deemed to have converged. The \( x \)- and \( y \)-components of the transverse part \( \mathbf{E}_T \) of the electric field are then computed by direct differentiation of expression (4), from which the transverse part of the magnetic induction field may be obtained easily using equation (2).

3. Presentation of numerical results. To begin this section, it is instructive to consider a resonator from which the shields have been removed; this can be modeled easily here by deleting from equations (12) and (17) all the terms involving the densities \( j_{l,k}^S \). Typical current densities for the rungs of the primary coil are shown in Figure 3 for a resonator with \( N = 8 \) conducting segments. The radius of the coil is \( a = 32 \text{ mm} \) and each rung on the primary subtends an angle of about \( 8.97^\circ \), so the results shown here are appropriate to a rather narrow-runged primary. (In fact, the
current density (in Amps/meter) for an 8-rung unshielded primary coil, as a function of angle $\theta$. The coil radius is $a = 32$ mm. Inter-rung feeding of the RF signal onto the primary circuit between rungs 2 and 3 has been employed.

In addition, the currents $I_{x,m}$ for the eight rungs are 3.827, 9.238, 9.238, 3.827, $-3.827$, $-9.238$, $-9.238$, and $-3.827$ mA. This corresponds to a situation in which the RF current is fed onto the primary circuit at points between the rungs, since it has been observed both numerically and experimentally by Crozier et al. [8] that this approach gives better homogeneity in the magnetic field within the coil. In this example, the current is fed between rungs 2 and 3 (see Figure 2).

The results in Figure 3 are in excellent agreement with those obtained in Crozier et al., where a conformal mapping method was used, and only six iterations are required here to achieve a converged solution. It is evident that the current densities $j_{l,m}$, $m = 1, 2, \ldots, 8$, become infinite at each edge of the conducting rungs, which is consistent with the work of Carlson [6] and Forbes, Crozier, and Doddrell [10]. This is, of course, to be expected, since it is known that square-root--type singularities are possible at these points, as is clear from the form of the solution (10). Spurious numerical oscillations, associated with the fact that the system of equations (7) is ill posed, have been completely eliminated here by means of the inverse finite Hilbert transform used in (12), and this is evident from the high quality of the results shown in Figure 3.

A contour map is shown in Figure 4(a) for the magnitude $\sqrt{\mathbf{E} \cdot \mathbf{E}} = \sqrt{\mathbf{E}_T \cdot \mathbf{E}_T}$ of the electric field vector produced by the current densities in Figure 3, for a coil of radius $a = 32$ mm. (It is straightforward to show, using (2) and the properties of the scalar triple product, that the magnitude of the magnetic induction vector $\mathbf{B}$ is the same function, except for a multiplicative constant.) The coil clearly produces a homogeneous region out to a radius of about 20 mm, but beyond this distance from the center, nonuniformities in the field become apparent.
These features are highlighted in Figure 4(b), which is a three-dimensional view of the electric field produced inside and outside the radius $a = 32$ mm, at which the conducting rungs are placed. For ease of viewing, field strengths greater than 150 volts/meter are not shown so that the homogeneous portion of the field close to the center may be more visible. The gradual fall in field strength outside the primary coil may also be seen from this figure.

The effects of placing a shield outside the coil studied in Figures 3 and 4 are now investigated. A narrow-runged primary is used, as before, with inter-rung feeding of
Current density (in Amps/meter) for a resonator with 8 primary rungs and 8 shielding rungs. The primary rungs (current densities sketched with solid lines) are placed at radius $a = 32$ mm, and the shielding rungs (dashed lines) are at radius $b = 42$ mm. Inter-rung feeding of the RF signal onto the primary circuit has been employed.

The signal onto the primary, as in Figure 3. Current densities are shown in Figure 5 for this case, in which there is a $1^\circ$ gap between each shielding segment; in practice, this has been found to be sufficient to reduce the interaction of pulsed magnetic field gradients. The radius of the primary coil is $a = 32$ mm, and the shields are located at radius $b = 42$ mm. The current densities for the primary coil (sketched with solid lines) are largely similar to those in Figure 3 for the unshielded case, although now the presence of the shields gives primary current densities that are somewhat more symmetric about the centers of each rung. For the results in Figure 5, the shielding rungs carry no nett current, so $I_k^S = 0$ for all eight shielding segments, $k = 1, 2, \ldots, 8$. Nevertheless, it is clear that small eddy currents are induced in each shielding rung by the RF signal on the primary, and the current densities associated with these are sketched with dashed lines in Figure 5.

A contour plot for this case is shown in Figure 6(a) and applies equally to the magnitude of either the electric or magnetic field, as explained in the discussion of Figure 4(a). By comparing this diagram with the equivalent unshielded coil case in Figure 4(a), it is evident that the homogeneous region in the center of the coil is actually enlarged by the presence of the shielding coils in Figure 6(a). In addition, the field outside the resonator decays more rapidly than in the unshielded case.

These features are confirmed by an examination of the three-dimensional representation of the magnitude of the electric field displayed in Figure 6(b). The homogeneous region within the resonator is visible on the diagram, and it is also clear that the field drops sharply outside the shielding coils over much of the circumference, although some field leakage still occurs, particularly near the gaps in the shields.

In the paper by Crozier et al. [8], the results for the case in which the RF signal was fed to a point in between the rungs of the primary circuit were compared with
results obtained when the signal was fed directly onto the rungs themselves, and it is appropriate to investigate this case here also, since this method of supplying the primary circuit is one which is used in practice. In Figure 7, the coil geometry for the shielded resonator is as before, except that the primary currents \( I_m \), \( m = 1, 2, \ldots, 8 \), now are 0.0, 7.071, 10.0, 7.701, 0.0, −7.701, −10.0, and −7.701 mA, which is appropriate to the RF current being fed onto rungs 1 and 5 in Figure 2. There is again no nett current flowing on any of the shielding segments.

From Figure 7, it may be seen that the current densities on rungs 2, 3, and 4 and 6, 7, and 8 are qualitatively similar to those obtained for the inter-rung-fed case shown in Figure 5; the square-root type singularities in \( j_t \) which are present at each edge of the rung give rise to the horseshoe-shaped nature of these diagrams.

**Fig. 6.** (a) Contour plot and (b) three-dimensional graph of the magnitude of the electric field vector for the shielded coil in Figure 5. Distances on the x and y axes are given in meters, and in 6(b), the vertical axis has been clipped at 150 volts/meter to aid viewing.
However, on rungs 1 and 5, for which there is no nett current flowing, it is evident that significant eddy currents are induced by the presence of the neighboring current-carrying rungs. In fact, current flowing down one edge of the rung is balanced by an equal and opposite current flowing up the other edge, as is required by symmetry. In addition, eddy currents are induced on the shielding rungs by the currents flowing in primary rungs 2, 3, and 4 and 6, 7, and 8 and are similar to those seen for the inter-rung-fed case in Figure 5.

The electric field produced in the rung-fed case by the current densities shown in Figure 7 is illustrated in Figure 8. This contour map indicates that the homogeneous region at the center of the coil is significantly smaller than for the inter-rung-fed case shown in Figure 6(a). This is essentially a consequence of the loss of symmetry in the field, with a clear axis formed through rungs 1 and 5. It appears that a better field is generated when there is a more even balance of nett current on the rungs of the primary coil, so the situation in which some rungs carry zero current while others take the maximum is avoided. Thus the strategy of feeding current into the circuit between the rungs is expected to be superior to a direct rung-fed approach, and this is confirmed by the observations of Crozier et al. [8].

As higher-frequency resonators are being developed, a need arises for primary circuits with wider rungs, in order to reduce the inductance of the device. In Figure 9, the inter-rung feeding strategy of Figure 5 has been applied to a resonator in which the rungs on the primary have been increased in width; here the angle subtended by the primary rungs is now 39.6°, with 1° gaps in the shields as before. It is evident that eddy current generation is now a major influence on the current distribution within each primary rung, since rungs 1, 4, 5, and 8 actually contain countercurrents,
in spite of the fact that the nett currents on these rungs are not zero. In addition, large eddy currents are present on the shielding segments, each of which carries no nett current.
(a)

(b)

Fig. 10. (a) Contour plot and (b) three-dimensional graph of the magnitude of the electric field vector for the wide-runged, inter-rung-fed shielded coil in Figure 9. Distances on the $x$ and $y$ axes are given in meters, and in (b), the vertical axis has been clipped at 150 volts/meter to aid viewing.

The field produced by this wide runged resonator is illustrated in Figures 10(a) and 10(b). The contour map of Figure 10(a) suggests that the region of homogeneity within the coil has been reduced slightly by the use of wide primary rungs, and this is consistent with the findings of Crozier et al. [8]. These features are further highlighted by the three-dimensional representation shown in Figure 10(b).

4. The quality factor. In this section, we derive formulae for the quality factor $Q$ of the resonator, which is defined to be

$$Q = \frac{\omega}{\text{energy stored in the resonator}} / \text{power lost in the resonator}.$$
Now electromagnetic energy density is given by the scalar quantity

\[
\frac{1}{2} (E \cdot D + H \cdot B),
\]

which follows on physical grounds from Poynting’s theorem. By making use of the constitutive relations \( D = \epsilon E \) and \( B = \mu H \), the TEM approximation (1) and the relation (2), and integrating over the cross section of the resonator, it follows that the energy stored per unit length within the coil is given by

\[
\mathcal{E} = \epsilon \int \mathbf{E}_T \cdot \mathbf{E}_T dS.
\]

It follows from the use of the TEM approximation that a scalar potential \( \Phi \) exists for the transverse part of the electric field, according to the relation \( E = -\nabla^2 \Phi \), as in section 2. Inserting this relation into equation (22), integrating by parts, and making use of the divergence theorem of Gauss then give

\[
\mathcal{E} = \epsilon \oint_{r=a} \Phi \frac{\partial \Phi}{\partial n} ds,
\]

where \( s \) denotes the arclength around the entire circumference \( r = a \) and \( n \) is the inward-pointing normal. After a little algebra, the energy stored per unit length of the resonator is obtained from (23) in the form

\[
\mathcal{E} = \epsilon a \int_0^{2\pi} \Phi(a, \theta) E_r(a, \theta) d\theta,
\]

in which \( E_r = -\partial \Phi/\partial r \) is the radial component of the electric field vector.

In order to make use of the formula (24) for the stored energy per unit length, it is necessary to compute both the scalar potential \( \Phi \) and its normal derivative on the circumference \( r = a \). From (5) we have

\[
\Phi(a, \theta) = -\frac{a}{\pi} \sqrt{\frac{\mu}{\epsilon}} \sum_{n=1}^{N} \int_{\theta_{n,1}}^{\theta_{n,2}} j_{i,n}^{p} (\beta_n) \ln \left| 2 \sin \left( \frac{\beta_n - \theta}{2} \right) \right| d\beta_n
\]

\[
-\frac{b}{\pi} \sqrt{\frac{\mu}{\epsilon}} \sum_{l=1}^{L} \int_{\psi_{1,1}}^{\psi_{1,2}} j_{i,1}^{S} (\psi_l) \ln \frac{1}{5} \sqrt{a^2 + b^2 - 2ab \cos (\psi_l - \theta)} d\psi_l,
\]

provided that \( \theta \) does not lie within any of the intervals \( \theta_{n,1} < \theta < \theta_{n,2} \); in other words, the formula (25a) is applicable between the primary conducting rungs on \( r = a \). When \( \theta = \theta_m \) describes a point on the \( m \)th conducting rung, then the first integral in the expression (25a) becomes singular for \( n = m \), and although the singularity is integrable, it is advisable for numerical purposes to remove it by subtraction. Thus, for a point on the \( m \)th conducting rung on the primary coil,
Figure 11 shows the potential function $\Phi(a, \theta)$ evaluated on the circle $r = a$, computed using (25). Results are shown for two different types of shielded resonator, and in both instances, the primary circuit has been fed between the rungs. The case sketched with the solid line corresponds to the narrow-runged coil of Figure 5, while the dashed line is the potential produced by the wide-runged resonator of Figure 9. Such a calculation actually serves as a sensitive check on the accuracy of our method, since the results in Figure 11 are the outcome of a numerical solution of the integral equations (12), combined with a synthesis of the scalar potential field using (25). The
boundary conditions indicate that $\Phi$ should be constant on each rung, and Figure 11 shows that this is the case, apart from a small drift caused by numerical error.

The formula (24) for the energy per unit length stored within the coil also requires a method for the calculation of the radial component $E_r$ of the transverse part of the electric field vector. This may be obtained without difficulty by direct differentiation of (5) in the limit $r \to a$ and making use of the definitions (11a). This yields

$$E_r(a, \theta) = \frac{1}{\pi} \sqrt{\frac{\mu}{\epsilon}} \left\{ \frac{1}{4a} \sum_{\ell=1}^{N} \int \frac{I_{\ell}^n}{J_{\ell}^n} + b \sum_{\ell=1}^{L} \int_{\theta_{\ell,1}}^{\theta_{\ell,2}} \frac{j^{2}_{\ell}(\psi)}{a^2 + b^2 - 2ab \cos(\psi - \theta)} \, d\psi \right\}. $$

(26)

The energy stored per unit length of resonator is thus computed using (24)–(26).

In order to estimate the power lost in the resonator, as required by formula (21) for the quality factor $Q$, it is necessary to account for ohmic losses within the conducting primary and shielding rungs in the device. This is done using a conductivity $\sigma$ and assuming that the current flows on both sides of the thin conductors in shallow surface layers of skin depth $\delta$. The ohmic power loss per unit length of the resonator is therefore

$$P = \frac{1}{\sigma \delta} \sum \oint |n \times H|^2 \, ds,$$

(27)

where the sum in this expression is taken over all the conducting rungs. By making use of the relation (2) and the properties of the vector triple product, the formula (27) may eventually be written as

$$P = \frac{2\epsilon}{\sigma \delta \mu} \left\{ a \sum_{n=1}^{N} \int_{\theta_{n,1}}^{\theta_{n,2}} [E_r(a, \theta)]^2 \, d\theta + b \sum_{\ell=1}^{L} \int_{\theta_{\ell,1}}^{\theta_{\ell,2}} [E_r(b, \theta)]^2 \, d\theta \right\}. $$

(28)

The evaluation of this formula (28) requires that an expression be developed for the function $E_r$ evaluated on the outer circle $r = b$, and this may be done exactly as in (26).

The quality factor, derived from (21) in the form $Q = \omega E / P$, has been evaluated for these resonators, assuming an operating frequency of 300 MHz, so that $\omega = 6\pi \times 10^8$ radians/sec. Using values of conductivity $\sigma \approx 6 \times 10^7$ mhos/meter and skin depth $\delta \approx 3.8 \times 10^{-6}$ meters, appropriate to copper at this frequency, values of $Q$ are typically found to be about 3700 for narrow-runged primary resonators and slightly lower ($Q \approx 3400$) in the wide-runged case. These high values for $Q$ are consistent with quality factors of microwave cavities [18].

In experimental measurements of quality factors for unloaded resonators of this type, however, values of the order of $Q \approx 200$ are typical (see Crozier et al. [8]). Clearly the ohmic losses within the rungs themselves are not the only sources of power dissipation in the system, and another must be accounted for in the model.

It turns out that a very significant source of power loss occurs within the capacitors connecting the rungs of the primary resonator at each end. A full model of the
frequency-dependent resistance of these devices is outside the scope of the present paper, but a simple direct-current (DC) estimate of power loss serves to make the point. The capacitors used in the probe shown in Figure 1 possessed an effective series resistance (ESR) at 300 MHz of $R_C \approx 0.15$ ohms (according to the manufacturer’s specifications), and for inter-rung feeding of a primary circuit with $N$ rungs, a simple analysis of the circuit diagram suggests that the DC system is equivalent to four parallel circuits, each containing $\frac{1}{2}N - 1$ of these resistors, with an extra resistor on each of two feeder circuits. The total DC resistance of such a unit is easily calculated to be

$$R_{\text{total}} = \frac{1}{4} \left( \frac{1}{2}N + 1 \right) R_C,$$

so that the DC power loss per unit length in the circuit is therefore

$$\frac{1}{4L} \left( \frac{1}{2}N + 1 \right) R_C \left( \frac{I_P^P}{\sqrt{2}} \right)^2,$$

where $L$ is the length of the primary coil and $I_P^P$ is the maximum current in the primary circuit. When (29) is added to (28), using values $N = 8$ rungs, $L \approx 7$ cm and $I_P^P = 10$ mA, quality factors of about $Q \approx 320$ are obtained.

As the quality factor $Q$ of the probe strongly affects the SNR of the NMR/MRI experiment and therefore the clarity of an image, it is clear that a major cause of image degradation is the power loss associated with the connecting capacitors. The alleviation of this problem is an area for future research.

5. Conclusions. In this paper, an accurate method has been presented for computing the current densities within the rungs of RF probes that are used in MRI. Because of the high frequency of the signals used in these devices, broad curved rungs are required instead of narrow wires, so it is not appropriate simply to assume a constant current density within the conducting rungs. There results a poorly conditioned system of integral equations to be solved for the current densities, given a knowledge of the nett (integrated) currents in each rung. In addition, it is known that the current density is predicted to approach infinity along the edges of each rung.

These difficulties have been overcome here, using a numerical method which explicitly accounts for the high current densities at the edges of conducting rungs. By applying the inverse finite Hilbert transform iteratively to each equation in the system, an accurate representation is obtained for the current densities both within the primary rungs and induced on the shielding segments of the resonator. Spurious numerical oscillations due to the poor conditioning of the system are completely eliminated by this approach.

Results have been shown for several coils of practical interest, and the fields computed within these devices are in substantial agreement with experimental observations [8]. The most homogeneous fields within the probe appear to be produced when the RF signal is fed onto the primary circuit between the rungs, and the presence of a shield in this case actually improves the field homogeneity by enhancing the symmetry of the current distributions. Wide-runged resonators have also been studied, but they produce less homogeneous fields and generate large eddy currents, as has been observed experimentally [8].

Quality factors for these resonators have been computed, and very high values are obtained, consistent with known results for microwave cavities. In practice, however,
losses in the system are responsible for much lower values of $Q$ being observed in the laboratory than these theoretical limits would suggest. We have indicated here that a major source of these losses is power dissipation within the capacitors in the circuitry for the primary coil, and this may suggest the possibility of future improvements.

Appendix: Transformation of some singular integrals. Consider the CPV integral $Y_3$ defined in (15). Making the change of variable

$$\beta = \frac{1}{2}(\theta_m,1 + \theta_m,2) + \frac{1}{2}(\theta_m,2 - \theta_m,1)t$$

converts this into the expression

$$Y_3(R_3, S_3, T_3) = \frac{1}{2} \text{CPV} \int_{-1}^{1} \sqrt{\frac{1 + t}{1 - t}} \cot \left( \frac{R_3 - S_3t}{2} \right) \frac{dt}{T_3 + t},$$

in which the arguments $R_3$, $S_3$, and $T_3$ are as defined in (13a). A further change of variable

$$w^2 = \frac{1 + t}{1 - t}$$

in the expression (A2) gives the result

$$Y_3(R_3, S_3, T_3) = 2 \text{CPV} \int_{0}^{\infty} \cot \left( \frac{R_3}{2} - \frac{S_3}{2} \frac{w^2 - 1}{w^2 + 1} \right) \frac{w^2 dw}{[w^2 + 1][T_3(w^2 + 1) + (w^2 - 1)]}.$$ 

Finally, the variable change

$$w = \tan \gamma$$

converts the expression (A4) into a trigonometric integral of the form

$$Y_3(R_3, S_3, T_3) = \text{CPV} \int_{0}^{\pi/2} \sin^2 \gamma \cot \left( \frac{1}{2} R_3 + \frac{1}{2} S_3 \cos 2\gamma \right) \frac{d\gamma}{(1 + T_3)/2 - \cos^2 \gamma}.$$

The integrand in (A6) has a pole singularity when $\cos \gamma = \cos \gamma_0 = \sqrt{(1 + T_3)/2}$, and this may be removed by subtraction, making use of the known result

$$\text{CPV} \int_{0}^{\pi/2} \frac{d\gamma}{\cos^2 \gamma_0 - \cos^2 \gamma} = 0.$$ 

Equation (A6) can therefore be written as the nonsingular integral

$$Y_3(R_3, S_3, T_3) = \int_{0}^{\pi/2} \left\{ \sin^2 \gamma \cot \left( \frac{1}{2} R_3 + \frac{1}{2} S_3 \cos 2\gamma \right) \right. 
- \left. \frac{1}{2} (1 - T_3) \cot \left( \frac{1}{2} R_3 + \frac{1}{2} S_3 T_3 \right) \right\} \frac{d\gamma}{(1 + T_3)/2 - \cos^2 \gamma},$$

which is well suited to numerical evaluation.
The transformation of the singular integral $Y_4$ in (16) follows exactly the same lines as above. The sequence of variable changes (A1), (A3), and (A5) is made to the original integral, and the singularity is then subtracted from the integrand and evaluated explicitly using the result (A7). The final nonsingular form of (16) is

\[ Y_4(R_4, S_3, T_3; a, b) = \int_0^{\pi/2} \left\{ \frac{ab \sin^2 \gamma \sin(R_4 + S_3 \cos 2\gamma)}{a^2 + b^2 - 2ab \cos(R_4 + S_3 \cos 2\gamma)} \right. \\
- \frac{(1/2)ab(1 - T_3) \sin(R_4 + S_3 T_3)}{a^2 + b^2 - 2ab \cos(R_4 + S_3 T_3)} \left. \right\} \frac{d\gamma}{(1 + T_3)/2 - \cos^2 \gamma}. \tag{A9} \]

Similarly, the expression (14) for $Y_1$ can be converted to the nonsingular form

\[ Y_1(R_3, S_3, T_3) = \int_0^{\pi/2} \left\{ \sin^2 \gamma \left( \cot \left[ \frac{1}{2} R_3 + \frac{1}{2} S_3 \cos 2\gamma \right] - \frac{2}{R_3 + S_3 \cos 2\gamma} \right) \right. \\
- \frac{1}{2}(1 - T_3) \left( \cot \left[ \frac{1}{2} R_3 + \frac{1}{2} S_3 T_3 \right] - \frac{2}{R_3 + S_3 T_3} \right) \left. \right\} \frac{d\gamma}{(1 + T_3)/2 - \cos^2 \gamma}. \tag{A10} \]

The nonsingular integrals (18)–(20) in the text are treated using the same transformations (A1), (A3), and (A5) as those that were used to obtain expressions (A8)–(A10). The integral $Y_2$ in (18) becomes

\[ Y_2(R_3, S_3) = \int_0^{\pi/2} \sin^2 \gamma \left( \cot \left[ \frac{1}{2} R_3 + \frac{1}{2} S_3 \cos 2\gamma \right] - \frac{2}{R_3 + S_3 \cos 2\gamma} \right) d\gamma, \tag{A11} \]

and $Y_5$ in (19) transforms to

\[ Y_5(R_3, S_3) = \int_0^{\pi/2} \sin^2 \gamma \cot \left[ \frac{1}{2} R_3 + \frac{1}{2} S_3 \cos 2\gamma \right] d\gamma. \tag{A12} \]

Equation (20) takes the form

\[ Y_6(R_4, S_3; a, b) = \int_0^{\pi/2} \frac{ab \sin^2 \gamma \sin(R_4 + S_3 \cos 2\gamma)}{a^2 + b^2 - 2ab \cos(R_4 + S_3 \cos 2\gamma)} d\gamma. \tag{A13} \]

These expressions (A8)–(A13) are evaluated easily and accurately using trapezoidal rule integration.

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