

On buckling of granular columns with shear interaction: Discrete versus nonlocal approaches

Noël Challamel¹, Jean Lerbet², and C. M. Wang³

Citation: *J. Appl. Phys.* **115**, 234902 (2014); doi: 10.1063/1.4883540

View online: <http://dx.doi.org/10.1063/1.4883540>

View Table of Contents: <http://aip.scitation.org/toc/jap/115/23>

Published by the [American Institute of Physics](#)

On buckling of granular columns with shear interaction: Discrete versus nonlocal approaches

Noël Challamel,^{1,a)} Jean Lerbet,^{2,b)} and C. M. Wang^{3,c)}

¹Université de Bretagne Sud, EA 4250, LIMATB, (Laboratoire d'Ingénierie des MATériaux de Bretagne), F-56100 Lorient, France

²Université d'Evry Val d'Essonne, UFR Sciences et Technologie40, rue du Pelvoux CE 1455, 91020 Evry Courcouronnes Cedex – France

³Engineering Science Programme and Department of Civil and Environmental Engineering, National University of Singapore, Kent Ridge, Singapore 119260

(Received 25 April 2014; accepted 2 June 2014; published online 16 June 2014)

This paper investigates the macroscopic behaviour of an axially loaded discrete granular system from a stability perspective. The granular system comprises uniform grains that are elastically connected with some bending and shear interactions and confined by some elastic supports. This structural system can then be classified as a discrete repetitive system, a lattice elastic model or a Cosserat chain model. It is shown that this Cosserat chain model is exactly tantamount to the finite difference formulation of a shear-deformable Timoshenko column in interaction with a Winkler foundation. The buckling of the discrete column with pinned ends is first analytically investigated through the resolution of a finite difference equation. The solution is compared to a nonlocal approach derived by continualizing the discrete problem. The approximated Timoshenko nonlocal approach appears to be efficient with respect to the reference lattice problem and highlights some specific scale effects. This scale effect is related to the grain size with respect to the total length of the Cosserat chain. Finally, the paper shows the key role played by the shear interaction in the instabilities of granular structural system, especially when the bending interaction can be neglected. © 2014 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4883540>]

I. INTRODUCTION

Granular materials are involved in a large spectrum of engineering applications from pharmaceutical engineering, food engineering to civil engineering. These apparently simple materials are reputed to show extremely complex behaviors with the coexistence of different kinds of macroscopic behavior including both its granular aspect with strong inelastic nonlinear constitutive laws.¹ Among the specificities of its discrete or granular character, the existence of force chains that may buckle still remains an important scientific task to be investigated.^{2,3} One can say that structural mechanics has recently found a renewed interest in this small microscopic scale in order to explain some fundamental behaviors especially when analyzing the instability phenomena of hierarchical grain structures along with force chains and force transmission. Most studies in this field focus on numerical investigations of these discrete elastic or inelastic chains by using the so-called Discrete Element Method.^{2,4} Hunt *et al.*⁵ presented some numerical results for the linearized buckling behaviour of a finite elastic shear-bending chain (which can also be labeled as a Cosserat chain). They also investigated the post-buckling behavior with stable and unstable bifurcation branches. The fundamental properties of this finite-size chain are rather difficult to characterize in an exhaustive way within these numerical

performance and a complementary approach that should be able to link the discrete problem with an enriched continuum model would certainly be decisive in forming a better understanding of the main phenomena in granular instabilities.

In this paper, we show that the buckling of a discrete shear deformable column can be investigated analytically; thereby leading to a more tractable parametric sensitivity study. Pasternak and Mühlhaus⁶ also presented some analytical results for the bending of a finite size discrete shear system, and highlighted the link with nonlocal elastic media. Luongo and Zulli⁷ studied the dynamics of a finite size discrete shear system for specific applications in the vibrations of shear buildings. They obtained the exact solution of the natural frequencies of the equivalent lattice shear system. More recently, Zhang *et al.*⁸ or Duan *et al.*⁹ obtained some analytical results for the buckling and the vibrations of discrete shear repetitive systems. They show the link between discrete shear systems and nonlocal Timoshenko continuous theories for beam-like structures. These results can be viewed as the generalization of discrete bending systems, as already characterized in bending, vibrations and buckling.^{10–13} Discrete or finite-size microstructured systems are shown to behave as nonlocal structural systems, thus explaining the specific scale effect that may be predominant for micro and nanostructures. The equivalent nonlocality may be classified for these systems as an Eringen's type nonlocality¹⁴ with a single additional length scale. As we will see in this paper, the same phenomenon may control the instability phenomenon in granular chains.

^{a)}E-mail: noel.challamel@univ-ubs.fr.

^{b)}Email: jean.lerbet@ibisc.univ-evry.fr.

^{c)}E-mail: ceewcm@nus.edu.sg.

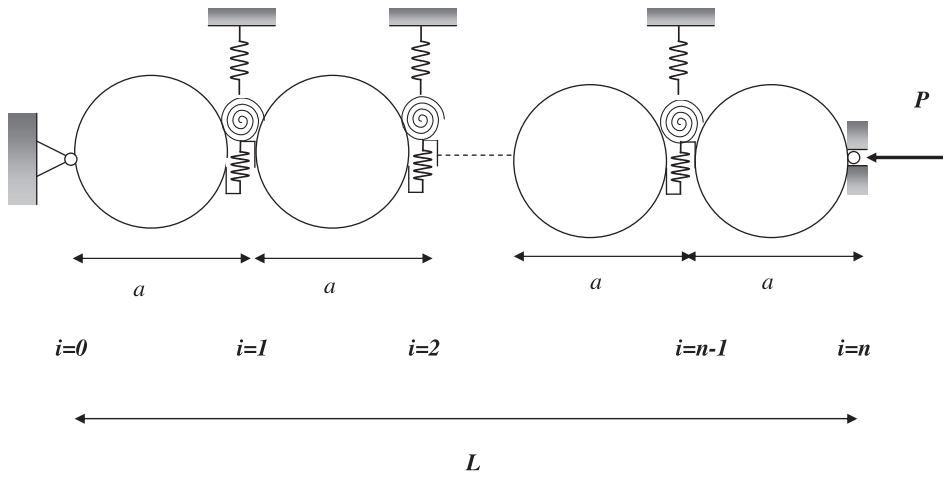


FIG. 1. A discrete shear granular chain composed of n grains of diameter a ; $L = n \times a$.

This paper investigates the macroscopic behaviour of a granular system from a stability perspective. The granular system is composed of uniform grains that are elastically connected. This structural system can then be classified as a discrete repetitive system or a lattice model, as introduced by Born and von Kármán in 1912 for infinite axial chain.¹⁵ The discrete bending chain is generally referred to the Hencky chain model,¹⁶ which can be generalized, as investigated in this paper, considering both shear and bending interaction between each cell. In fact, the granular system considered herein comprises two independent degrees of freedom (i.e., the displacement and the rotation), which can then be regarded as a Cosserat-type chain.^{6,17} Its beam continuum analogy is the Timoshenko model.^{18,19} It can be shown that the one dimensional shear and rotational lattice model considered by Ostoja-Starzewski²⁰ is also exactly the microstructured Timoshenko model that we considered in our paper (see also recently Ref. 21 for the vibrations of Timoshenko beam elements analyses by a lattice shear system). As a consequence, the physical system considered herein is a granular chain, but this lattice model can be also referred to a Cosserat chain or a discrete Timoshenko structural system. This structural system can also be considered as a one-dimensional micropolar system.

II. SHEAR GRANULAR SYSTEM—A DISCRETE APPROACH

Consider the microstructured granular chain comprising n rigid grains (whose diameter $a = L/n$) that are connected by $n + 1$ rotational and shear springs (see Figure 1). This discrete grain column is subjected to a compressive axial load P . The problem at hand is to determine the buckling load of this discrete shear system for archetypal boundary conditions such as simply supported ends.

At each node i , there are two degrees of freedom, representing the nodal transverse displacement w_i of grain i (measured at the left-hand side of the connection between the grain i and the grain $i + 1$) and nodal rotation θ_i of grain i . The grain i is connected with the grain $i + 1$ at the node i as shown in Figure 2. A linearized buckling analysis will be presented based on energy arguments.

The strain energy function due to deformed rotational springs (bending term) is given as

$$V_b = \sum_{i=1}^{n-1} \frac{1}{2} C (\theta_{i+1} - \theta_i)^2, \quad (1)$$

where $C = \frac{nEI}{L} = \frac{EI}{a}$ and $L = na$. The quantity n is the number of grains (or discrete elements), a is the diameter of

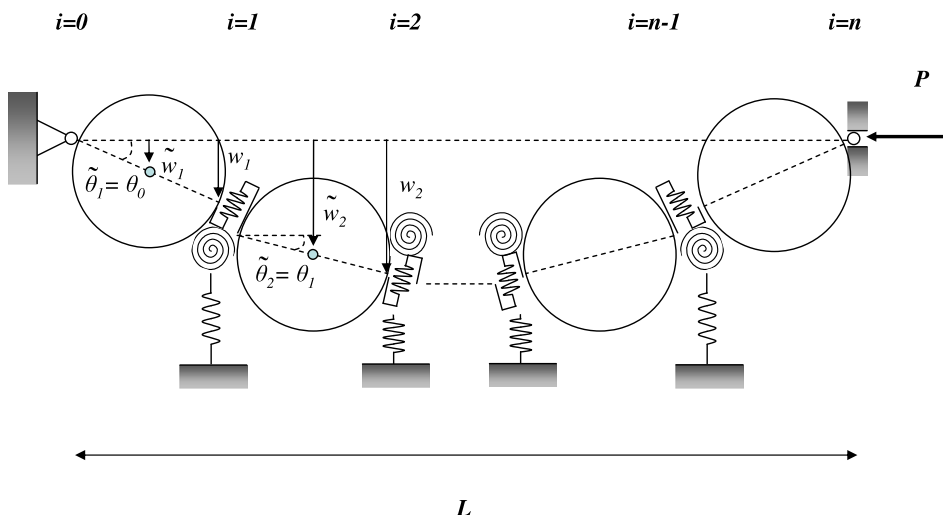


FIG. 2. The granular chain in deformed configuration.

each grain and L is the total length of the granular column. C is the rotational stiffness located at the connection between each grain. This discrete stiffness can be expressed with respect to the bending stiffness EI of the equivalent beam.

The strain energy function due to deformed shear spring (shear term) is given by

$$V_s = \sum_{i=1}^{n-1} \frac{1}{2} S (w_{i+1} - w_i - a\theta_i)^2, \quad (2)$$

where $S = \frac{\kappa GA}{L} = \frac{\kappa GA}{a}$, S is the shear stiffness which can be expressed with respect to the shear stiffness κGA of the equivalent beam.

In the present formulation, the kinematics variables are measured at nodes i located at the connexion between each grain. If instead, the kinematics variables would have been defined with respect to the center of each grain, one would obtain for the energy of the deformed shear spring (see also Ref. 6):

$$V_s = \sum_{i=1}^{n-1} \frac{1}{2} S \left(\tilde{w}_{i+1} - \tilde{w}_i - a \frac{\tilde{\theta}_i + \tilde{\theta}_{i+1}}{2} \right)^2, \quad (3)$$

where \tilde{w}_i now denotes the deflection of the center of the i -th grain, and $\tilde{\theta}_i$ the rotation of the center of the i -th grain. The relation between both equivalent models is obtained from:

$$\tilde{w}_i = w_i - \frac{a}{2} \tilde{\theta}_i \quad \text{and} \quad \tilde{\theta}_i = \theta_{i-1}. \quad (4)$$

With this change of variable (see also the geometrical interpretation of Figure 2), the energy function of the deformed shear spring given in Eq. (2) is equivalent to the one given in Eq. (3). In the sequel, one will keep the notation of Eq. (2), as conformed to the lattice equations of lattice shear systems.^{8,9,20}

For granular applications, the shear energy term is probably predominant compared to the bending term, as the rotational link can be probably neglected. An asymptotic expansion will be performed at the end of the paper, for specific applications to shear granular phenomena. The elastic energy in the discrete elastic support (discrete Winkler foundation²²) is given by

$$V_{Winkler} = \sum_{i=1}^{n-1} \frac{1}{2} K w_i^2, \quad (5)$$

where $K = ka$ is the discrete stiffness of the elastic support.

The work done by the compressive axial load on the discrete shear column is given by:

$$W = \frac{1}{2} \sum_{i=1}^{n-1} P a \left(\frac{w_{i+1} - w_i}{a} \right)^2. \quad (6)$$

Therefore, the total potential energy function can be expressed as

$$\begin{aligned} \Pi = & \frac{1}{2} \sum_{i=1}^{n-1} C (\theta_{i+1} - \theta_i)^2 + \frac{1}{2} \sum_{i=1}^{n-1} S (w_{i+1} - w_i - a\theta_i)^2 \\ & + \frac{1}{2} \sum_{i=1}^{n-1} K w_i^2 - \frac{1}{2} \sum_{i=1}^{n-1} P a \left(\frac{w_{i+1} - w_i}{a} \right)^2. \end{aligned} \quad (7)$$

The Euler-Lagrange equations based on the energy function in Eq. (7) are

$$\kappa GA \left(\frac{w_{i+1} - w_i}{a} - \theta_i \right) + EI \frac{\theta_{i+1} - 2\theta_i + \theta_{i-1}}{a^2} = 0, \quad (8a)$$

$$\begin{aligned} \kappa GA \left(\frac{w_{i+1} - 2w_i + w_{i-1}}{a^2} - \frac{\theta_i - \theta_{i-1}}{a} \right) \\ - P \frac{w_{i+1} - 2w_i + w_{i-1}}{a^2} - k w_i = 0. \end{aligned} \quad (8b)$$

One recognizes the finite difference equation for the buckling problem of a Timoshenko continuous column on Winkler foundation which is governed by the following ordinary differential equations:

$$\kappa GA (w' - \theta) + EI \theta'' = 0, \quad (9a)$$

$$\kappa GA (w'' - \theta') - P w'' - k w = 0. \quad (9b)$$

Returning to the discrete system Eq. (8), it is possible to extract the finite rotation θ_i , from:

$$\begin{aligned} EI \frac{\theta_{i+1} - 3\theta_i + 3\theta_{i-1} - \theta_{i-2}}{a^3} \\ + P \frac{w_{i+1} - 2w_i + w_{i-1}}{a^2} + k w_i = 0. \end{aligned} \quad (10)$$

The fourth-order finite difference equation is then obtained for the deflection:

$$\begin{aligned} \left(EI - P \frac{EI}{\kappa GA} \right) \frac{w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2}}{a^4} \\ + \left(P - k \frac{EI}{\kappa GA} \right) \frac{w_{i+1} - 2w_i + w_{i-1}}{a^2} + k w_i = 0. \end{aligned} \quad (11)$$

Equation (11) is the finite difference equation of the Timoshenko beam model in interaction with a Winkler foundation, and is governed by the following ‘‘local’’ differential equation:

$$\left(EI - P \frac{EI}{\kappa GA} \right) w^{(4)} + \left(P - k \frac{EI}{\kappa GA} \right) w'' + k w = 0. \quad (12)$$

By introducing the following dimensionless parameters,

$$\beta = \frac{PL^2}{EI}, \quad k^* = \frac{kL^4}{EI} \quad \text{and} \quad s^2 = \frac{EI}{\kappa GAL^2}, \quad (13)$$

the fourth-order finite difference equation may be rewritten as

$$\begin{aligned} w_{i+2} - 4w_{i+1} + 6w_i - 4w_{i-1} + w_{i-2} + \frac{1}{n^2} \frac{\beta - k^* s^2}{1 - \beta s^2} \\ \times (w_{i+1} - 2w_i + w_{i-1}) + \frac{1}{n^4} \frac{k^*}{1 - \beta s^2} w_i = 0. \end{aligned} \quad (14)$$

The buckling mode can be obtained from $w_i = B\lambda^i$, which by substitution into Eq. (14) leads to the characteristic equation:

$$\left(\frac{1}{\lambda} + \lambda\right)^2 + \left(\frac{1}{\lambda} + \lambda\right) \left(-4 + \frac{1}{n^2} \frac{\beta - k^* s^2}{1 - \beta s^2}\right) + 4 - \frac{2}{n^2} \frac{\beta - k^* s^2}{1 - \beta s^2} + \frac{1}{n^4} \frac{k^*}{1 - \beta s^2} = 0. \tag{15}$$

The characteristic equation admits four solutions obtained from the second-order polynomial equation:

$$\frac{1}{\lambda} + \lambda = 2 - \frac{1}{2n^2} \frac{\beta - k^* s^2}{1 - \beta s^2} \pm \frac{1}{2n^2} \times \frac{\sqrt{(\beta - k^* s^2)^2 - 4k^*(1 - \beta s^2)}}{1 - \beta s^2}. \tag{16}$$

These solutions may be expressed in the following form:

$$\begin{aligned} \lambda_{1,2} &= \cos \varphi \pm j \sin \varphi \quad \text{and} \\ \lambda_{3,4} &= 2 - \frac{1}{2n^2} \frac{\beta - k^* s^2}{1 - \beta s^2} - \cos \varphi \pm \sqrt{\left(2 - \frac{1}{2n^2} \frac{\beta - k^* s^2}{1 - \beta s^2} - \cos \varphi\right)^2 - 1} \quad \text{with} \\ \varphi &= \arccos \left(1 - \frac{1}{4n^2} \frac{\beta - k^* s^2}{1 - \beta s^2} - \frac{1}{2n^2} \sqrt{\left(\frac{\beta - k^* s^2}{2(1 - \beta s^2)}\right)^2 - \frac{k^*}{1 - \beta s^2}}\right), \end{aligned} \tag{17}$$

where $j^2 = -1$. The buckling mode for the case of pinned ends is obtained from the trigonometric-based solution related to the first roots $\lambda_{1,2}$, which are associated with the non-trivial condition:

$$\sin(n\varphi) = 0 \quad \Rightarrow \quad \varphi = \frac{p\pi}{n}, \tag{18}$$

where p is the mode number (natural number) which depends on the dimensionless confinement modulus k^* .

We finally obtain the buckling load of the discrete system as

$$\begin{aligned} \cos\left(\frac{p\pi}{n}\right) &= 1 - \frac{1}{4n^2} \frac{\beta - k^* s^2}{1 - \beta s^2} \\ &\quad - \frac{1}{2n^2} \sqrt{\left(\frac{\beta - k^* s^2}{2(1 - \beta s^2)}\right)^2 - \frac{k^*}{1 - \beta s^2}}. \end{aligned} \tag{19}$$

The dimensionless buckling load β can be extracted from this last equation, which leads to

$$\beta_n = \min_p \left\{ \frac{16n^4 \sin^4\left(\frac{p\pi}{2n}\right) + k^* \left[1 + 4s^2 n^2 \sin^2\left(\frac{p\pi}{2n}\right)\right]}{4n^2 \sin^2\left(\frac{p\pi}{2n}\right) + 16s^2 n^4 \sin^4\left(\frac{p\pi}{2n}\right)} \right\}. \tag{20}$$

In the absence of the Winkler foundation, i.e., for $k^* = 0$, the buckling load of the discrete granular column is simplified and is calculated from the fundamental mode $p = 1$:

$$\beta_n = \frac{4n^2 \sin^2\left(\frac{\pi}{2n}\right)}{1 + 4s^2 n^2 \sin^2\left(\frac{\pi}{2n}\right)}. \tag{21}$$

The above mentioned solution has been reported by Zhang *et al.* for the buckling of discrete shear deformable columns with pinned ends.⁸ An asymptotic expansion of this last buckling value (without any additional Winkler foundation) with respect to the number of grains furnishes

$$\beta_n = \frac{\pi^2}{1 + \pi^2 s^2} \left[1 - \frac{\pi^2}{12n^2(1 + \pi^2 s^2)}\right] + o\left(\frac{1}{n^4}\right). \tag{22}$$

One recognizes in the first term the buckling value of the ‘‘local’’ Engesser’s column^{23,24} which is valid for a sufficiently large number of grains:

$$\lim_{n \rightarrow \infty} \beta_n = \frac{\pi^2}{1 + \pi^2 s^2}. \tag{23}$$

The effect of additional lateral elastic constraint on the buckling load of the ‘‘local’’ shear Timoshenko column is obtained directly from Eq. (20) for a sufficiently large number of grains:

$$\lim_{n \rightarrow \infty} \beta_n = \min_p \left\{ \frac{p^2 \pi^2}{1 + s^2 p^2 \pi^2} + \frac{k^*}{p^2 \pi^2} \right\}. \tag{24}$$

It is easy to check that this formulae leads to the usual buckling load of the Euler-Bernoulli column on Winkler model in case of negligible shear effects $s^2 = 0$ (see, for instance, Refs. 24–31 for the Winkler problem). This buckling load is also the one obtained by Cheng and Pantelides³² or Wang *et al.*³³ for Engesser Timoshenko column on Winkler foundation, when the effect of transverse shear deformation is included.

Figure 3 shows the effect of the number of grains on the dimensionless buckling load for a shear inextensible column

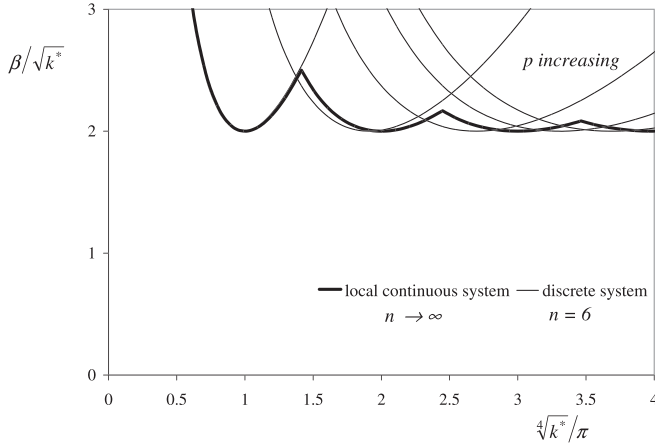


FIG. 3. Effect of number of grains n on dimensionless buckling load β of shear granular column; $n=6$; Scale effect induced by size ratio between diameter a of the grain with respect to total length of column L ; $s^2=0$.

$\kappa GA \rightarrow \infty$ or $s^2 \rightarrow 0$ (discrete system with bending interaction only). The finite dimensional problem may have a lower buckling load than its continuous “local” counterpart; a property that has been already analysed for other discrete systems that are modelled within nonlocal elasticity (see, for instance, Ref. 12). However, it is also observed that the finite dimensional system may have a larger buckling load than its continuous “local” counterpart, for some specific modulus of the stiffness restraints. The shear granular column clearly exhibits some scale effects, as the buckling diagram is affected by the number of grains, as opposite to the so-called “local” solution.

The pure shear discrete system (bending inextensible column) has been also investigated from Eq. (20) which can be asymptotically expanded as:

$$\beta_{shear} = \lim_{s \rightarrow \infty} \beta_n = \min_p \left\{ \frac{k^*}{4n^2 \sin^2 \left(\frac{p\pi}{2n} \right)} \right\}. \quad (25)$$

For the “local” continuous case, it is found that the buckling load of the continuous shear column can be neglected, which means that this shear configuration cannot be stable (no elastic stability domain):

$$\lim_{n \rightarrow \infty} \beta_{shear} = \min_p \left\{ \frac{k^*}{p^2 \pi^2} \right\} = 0. \quad (26)$$

However, the remarkable result is that this fundamental property is no longer true for the finite dimensional system. For the finite shear system, the buckling load can be calculated from

$$\beta_{shear} = \min_p \left\{ \frac{k^*}{4n^2 \sin^2 \left(\frac{p\pi}{2n} \right)} \right\} = \frac{k^*}{4n^2}. \quad (27)$$

Hence, we have solved a kind of paradox, where the continuous shear system cannot buckle (with a negligible shear stability domain), whereas the discrete shear system may present some specific structural instabilities above a critical

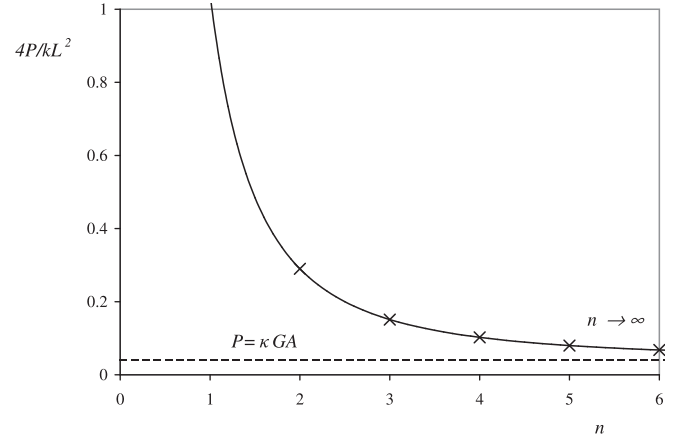


FIG. 4. Discrete shear column may buckle whereas continuous column (for $n \rightarrow \infty$) may not; $\frac{4P}{\kappa L^2} = \frac{4}{k^*} + \frac{1}{n^2}$; $k^* = \frac{\kappa L^2}{\kappa GA} = 100$.

value of the axial load. This apparent paradox is clearly explained in Figure 4.

III. SHEAR GRANULAR SYSTEM—A NONLOCAL APPROACH

The discrete two-field equations are extended to an equivalent continuum via a continualization method. The following relation between the discrete and the equivalent continuous system $w_i = w(x = ia)$ holds for a sufficiently smooth deflection function as:

$$w(x+a) = \sum_{k=0}^{\infty} \frac{a^k \partial_x^k w(x)}{k!} = e^{a \partial_x} w(x) \quad \text{with} \quad \partial_x = \frac{\partial}{\partial x}. \quad (28)$$

The discrete equations (11) are now continualized from this asymptotic expansion:

$$\left(EI - P \frac{EI}{\kappa GA} \right) \partial_x^4 w + \left(P - k \frac{EI}{\kappa GA} \right) \left(1 - \frac{a^2}{12} \partial_x^2 \right) \times \partial_x^2 w + k \left(1 - \frac{a^2}{6} \partial_x^2 \right) w + o(a^4) = 0. \quad (29)$$

The fourth-order differential equation can be re-expressed in the following way, where the fourth-order truncation in the asymptotic expansion is now omitted:

$$\left[EI \left(1 - \frac{P}{\kappa GA} \right) - l_c^2 \left(P - \frac{kEI}{\kappa GA} \right) \right] w^{(4)} + \left(P - k \frac{EI}{\kappa GA} - 2kl_c^2 \right) w'' + kw = 0 \quad \text{with} \quad l_c^2 = \frac{a^2}{12}. \quad (30)$$

The nonlocal continualized solution of Zhang *et al.*⁸ is recovered in the absence of the Winkler foundation, i.e.,

$$k = 0 \Rightarrow \left[EI \left(1 - \frac{P}{\kappa GA} \right) - Pl_c^2 \right] w^{(4)} + Pw'' = 0 \quad \text{with} \quad l_c^2 = \frac{a^2}{12}. \quad (31)$$

In this last case, the equations are the same as the ones of a nonlocal Timoshenko column based on Eringen’s nonlocality and obtained from the nonlocal constitutive law:

$$M - l_c^2 M'' = EI\theta' \quad \text{and} \quad V = \kappa GA(w' - \theta). \quad (32)$$

Note that the nonlocality here affects the bending part of the constitutive law and not the shear part.³⁴ The equilibrium equations are not affected by some nonlocal terms here:

$$M' + V = 0 \quad \text{and} \quad -V' + Pw'' = 0. \quad (33)$$

It can be shown that Eqs. (32) and (33) lead to the continualized differential equations Eq. (31) of the shear discrete Timoshenko column (as already shown by Zhang *et al.*⁸). The fully coupled nonlocal shear beam model considered by Reddy and Pang³⁵ and based on:

$$M - l_c^2 M'' = EI\theta' \quad \text{and} \quad V - l_c^2 V'' = \kappa GA(w' - \theta) \quad (34)$$

would lead to a slightly different differential equation:

$$EI \frac{Pl_c^2}{\kappa GA} w^{(6)} + \left[EI \left(1 - \frac{P}{\kappa GA} \right) - Pl_c^2 \right] w^{(4)} + Pw'' = 0, \quad (35)$$

where the sixth-order term is not present in the continualization approach.

The discrete shear model is thus equivalent to an Eringen nonlocal Timoshenko column, where the nonlocality is only affecting the bending part of the elastic constitutive law. Now considering the discrete shear model on discrete Winkler foundation, it can be shown that the continualized equations are the ones of a nonlocal Timoshenko column on a kind of Pasternak foundation, as detailed below for the enriched equilibrium equations:

$$M' + V = 0 \quad \text{and} \quad -V' + Pw'' + kw - kl_c^2 w'' = 0. \quad (36)$$

For a consistent analysis, as detailed for the discrete shear column without discrete Winkler foundation, we assume a nonlocal Eringen’s law affecting only the bending part of the constitutive law, as considered in Eq. (32). Combining Eq. (36) with Eq. (32) leads to the fourth-order differential equation:

$$\left[EI \left(1 - \frac{P}{\kappa GA} \right) - l_c^2 \left(P - \frac{kEI}{\kappa GA} \right) - kl_c^4 \right] w^{(4)} + \left(P - k \frac{EI}{\kappa GA} - 2kl_c^2 \right) w'' + kw = 0, \quad (37)$$

which is identical to Eq. (30) if the terms in l_c^4 are omitted. Hence, the discrete nature of the elastic restraint leads to a kind of nonlocal interaction, as investigated by Challamel *et al.*³¹ from the Pasternak and the Reissner model. It is shown herein that the nonlocal soil interaction may also come from the discreteness of the support.

Going back to the exact continualized model (which is equivalent to a nonlocal Eringen’s column supported by a

nonlocal foundation), which is an approximation of the discrete model, the consideration of the exact buckling mode for the pinned-pinned column in Eq. (30) and based on:

$$w = w_0 \sin\left(\frac{p\pi x}{L}\right) \quad (38)$$

leads to the approximated buckling load of the nonlocal column

$$P = \frac{k + k \left(\frac{p\pi}{L}\right)^2 \left(\frac{EI}{\kappa GA} + 2l_c^2\right) + \left(\frac{p\pi}{L}\right)^4 EI \left(1 + \frac{kl_c^2}{\kappa GA}\right)}{\left(\frac{p\pi}{L}\right)^2 + \left(\frac{p\pi}{L}\right)^4 \left(l_c^2 + \frac{EI}{\kappa GA}\right)} \quad \text{with} \quad l_c^2 = \frac{a^2}{12}. \quad (39)$$

The dimensionless buckling load β may be expressed in dimensionless variables as

$$\beta_n = \min_p \left\{ \frac{(p\pi)^4 + k^* \left[1 + (p\pi)^2 \left(s^2 + \frac{1}{6n^2} + \frac{(p\pi)^2 s^2}{12n^2} \right) \right]}{(p\pi)^2 + (p\pi)^4 \left(s^2 + \frac{1}{12n^2} \right)} \right\}. \quad (40)$$

Equation (24) is found again as a particular case for a large number of grains ($n \rightarrow \infty$). The Euler-Bernoulli continualized nonlocal beam is obtained for a vanishing value of s^2 (pure bending system without any shear links):

$$s^2 = 0 \Rightarrow \beta_n = \min_p \left\{ \frac{(p\pi)^4 + k^* \left[1 + \frac{(p\pi)^2}{6n^2} \right]}{(p\pi)^2 + \frac{(p\pi)^4}{12n^2}} \right\}. \quad (41)$$

The shear continualized nonlocal beam is asymptotically obtained for large value of s^2 ($s^2 \rightarrow \infty$, for pure shear system without any bending links):

$$s^2 \rightarrow \infty \Rightarrow \beta_n = \min_p \left\{ k^* \left(\frac{1}{12n^2} + \frac{1}{(p\pi)^2} \right) \right\} = \frac{k^*}{12n^2}. \quad (42)$$

The reason for the difference between the results issued of the exact discrete model Eq. (27) and the approximate solution Eq. (42) is due to:

$$\lim_{n \rightarrow \infty} \min_p \left\{ \frac{1}{4 \sin^2 \left(\frac{p\pi}{2n} \right)} \right\} = \frac{1}{4} \neq \min_p \lim_{n \rightarrow \infty} \left\{ \frac{1}{4 \sin^2 \left(\frac{p\pi}{2n} \right)} \right\} = +\infty. \quad (43)$$

The same analysis may be followed for the pure shear system from the dimensionless parameters:

$$\hat{\beta} = \frac{P}{\kappa GA} = \beta s^2, \hat{k}^* = \frac{kL^2}{\kappa GA} = k^* s^2 \quad \text{and} \quad s^2 = \frac{EI}{\kappa GAL^2}. \tag{44}$$

With these normalized shear parameters, the dimensionless buckling load associated with the continualized nonlocal model, is equal to:

$$\hat{\beta}_n = \min_p \left\{ \frac{(p\pi)^4 s^2 + \hat{k}^* \left[1 + (p\pi)^2 \left(s^2 + \frac{1}{6n^2} + \frac{(p\pi)^2 s^2}{12n^2} \right) \right]}{(p\pi)^2 + (p\pi)^4 \left(s^2 + \frac{1}{12n^2} \right)} \right\}. \tag{45}$$

The shear continualized nonlocal beam is asymptotically obtained for large value of s^2 ($s^2 \rightarrow \infty$, for pure shear system without any bending links):

$$\begin{aligned} s^2 \rightarrow \infty \Rightarrow \hat{\beta}_n &= \min_p \left\{ 1 + \hat{k}^* \left(\frac{1}{12n^2} + \frac{1}{(p\pi)^2} \right) \right\} \\ &= 1 + \frac{\hat{k}^*}{12n^2}, \end{aligned} \tag{46}$$

whereas the exact discrete system

$$\hat{\beta}_{shear} = \min_p \left\{ 1 + \frac{\hat{k}^*}{4n^2 \sin^2 \left(\frac{p\pi}{2n} \right)} \right\} = 1 + \frac{\hat{k}^*}{4n^2}. \tag{47}$$

This comparison between the discrete solution computed from Eq. (20) and the nonlocal approach given by Eq. (40) is shown in Figures 5 and 6. The nonlocal approach appears to be particularly efficient as a continuous approximation of the exact discrete solution and is specifically relevant for low values of stiffness modulus (confinement value). The discrepancy between the approximated nonlocal approach and the exact discrete solution increases with the stiffness of the interaction modulus k .

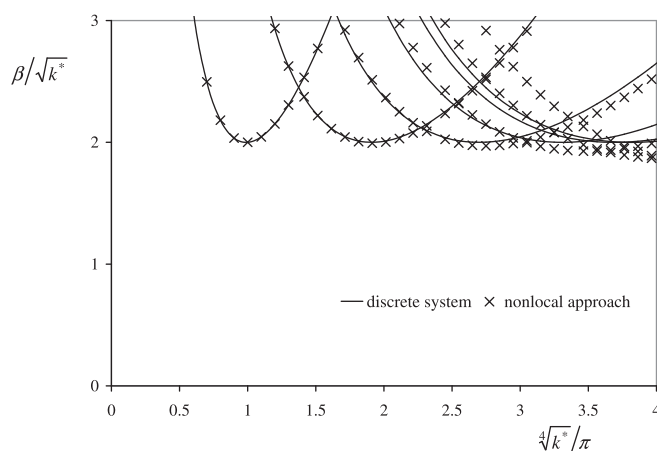


FIG. 5. Effect of number of grains n on dimensionless buckling load β of shear granular column; $n=6$; Comparison between exact discrete solution and nonlocal solution; $s^2=0$ (no shear interaction).

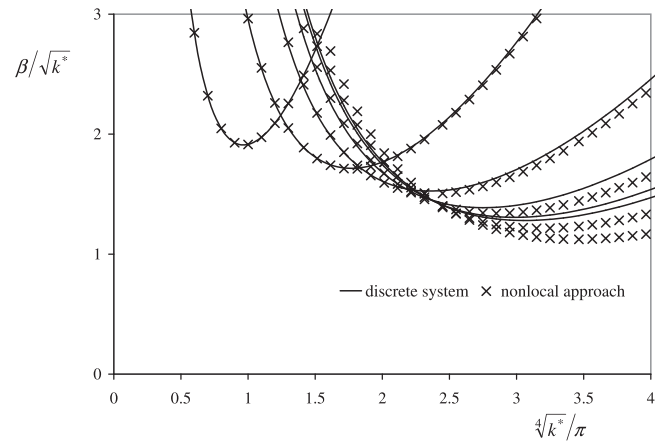


FIG. 6. Effect of number of grains n on dimensionless buckling load β of shear granular column; $n=6$; Comparison between exact discrete solution and nonlocal solution; $s^2=0.01$ (with shear interaction).

IV. CONCLUSIONS

This paper investigates the macroscopic behaviour of an axially loaded discrete granular system from a stability perspective. The granular system comprises uniform grains that are elastically connected with some bending and shear interactions and confined by some elastic supports. It is shown that this Cosserat chain model is exactly equivalent to the finite difference formulation of a shear deformable Timoshenko column with a Winkler foundation. The buckling of the discrete column with pinned ends is first analytically investigated through the resolution of a finite difference equation. It is shown for the pure shear system that the discrete granular column may buckle, whereas the continuous one cannot in the sense that the pure shear continuous system is unstable. This is a specific behaviour of the finite size structural system, which cannot be captured with a local continuous model.

The solution is compared to a nonlocal approach derived by continualizing the discrete problem. The approximated Timoshenko nonlocal approach appears to be efficient with respect to the reference lattice problem and highlights some specific scale effects. This scale effect is related to the grain size with respect to the total length of the Cosserat chain. Finally, the key role played by the shear interaction in the instabilities of granular structural system is revealed, especially when the bending interaction can be neglected. In the future, some more complex grain arrangements will be considered for a better understanding of the different kinds of behaviour that may appear in the instability of granular patterns.

ACKNOWLEDGMENTS

The authors thank F. Nicot and F. Darve for having pointed out the possible analysis of granular instabilities within elastic buckling theory.

¹F. Nicot and F. Darve, *Mech. Mater.* **43**, 918 (2011).

²A. Tordesillas and M. Muthuswamy, *J. Mech. Phys. Solids* **57**, 706 (2009).

- ³A. Tordesillas, D. M. Walker, and Q. Lin, *Phys. Rev. E* **81**, 011302 (2010).
- ⁴C. O. Sullivan, M. A. Wadee, K. J. Hanley, and D. Barreto, *Géotechnique* **63**, 1298–1309 (2013).
- ⁵G. W. Hunt, A. Tordesillas, S. C. Green, and J. Shi, *Philos. Trans. R. Soc. A* **368**, 249 (2010).
- ⁶E. Pasternak and H. B. Mühlhaus, *J. Eng. Math.* **52**, 199 (2005).
- ⁷A. Luongo and D. Zulli, *J. Sound Vib.* **330**, 3057 (2011).
- ⁸Z. Zhang, N. Challamel, and C. M. Wang, *J. Appl. Phys.* **114**, 114902 (2013).
- ⁹W. Duan, N. Challamel, C. M. Wang, and Z. Ding, *J. Appl. Phys.* **114**, 104312 (2013).
- ¹⁰N. Challamel, J. Lerbet, C. M. Wang, and Z. Zhang, *Z. Angew. Math. Mech.* **94**, 402 (2014).
- ¹¹N. Challamel, Z. Zhang, and C. M. Wang, “Nonlocal equivalent continua for buckling and vibration analyses of microstructured beams,” *J. Nanomech. Micromech.* doi:10.1061/(ASCE)NM.2153-5477.0000062 (published online 2013).
- ¹²N. Challamel, C. M. Wang, and I. Elishakoff, *Eur. J. Mech. A/Solids* **44**, 125 (2014).
- ¹³C. M. Wang, Z. Zhang, N. Challamel, and W. H. Duan, *J. Phys. D: Appl. Phys.* **46**, 345501 (2013).
- ¹⁴A. C. Eringen, *J. Appl. Phys.* **54**, 4703 (1983).
- ¹⁵M. Born and T. von Kármán, *Phys. Z.* **13**, 297 (1912).
- ¹⁶H. Hencky, “Habilitationsschrift,” D. Sc. thesis (Technical University of Darmstadt, Germany, 1920).
- ¹⁷E. Cosserat and F. Cosserat, *Théorie des corps déformables* (A. Herrmann et fils, Paris, 1909).
- ¹⁸S. P. Timoshenko, *Philos. Mag.* **41**, 744 (1921).
- ¹⁹S. P. Timoshenko, *Philos. Mag.* **43**, 125 (1922).
- ²⁰M. Ostoja-Starzewski, *Appl. Mech. Rev.* **55**, 35 (2002).
- ²¹M. Attar, A. Karrech, and K. Regenauer-Lieb, *J. Sound Vib.* **333**, 2359 (2014).
- ²²E. Winkler, *Die Lehre von der Elasticität und Festigkeit* (Prague, Dominicus, 1867).
- ²³F. Engesser, *Z. Arch. Ing.* **35**, 455 (1889).
- ²⁴S. P. Timoshenko and J. M. Gere, *Theory of Elastic Stability*, 2nd ed. (McGraw-Hill, Singapore, 1961).
- ²⁵M. Hetenyi, *Beams on Elastic Foundation* (Ann Arbor, The University of Michigan Press, 1946).
- ²⁶N. A. Alfutov, *Stability of Elastic Structures* (Springer-Verlag, Berlin, 2000).
- ²⁷Z. P. Bažant and L. Cedolin, *Stability of Structures—Elastic, Inelastic, Fracture, and Damage Theories* (Dover Publications, Inc., New York, 2003).
- ²⁸C. M. Wang, C. Y. Wang, and J. N. Reddy, *Exact Solutions for Buckling of Structural Members*, CRC Series in Computational Mechanics and Applied Analysis (Boca Raton, Florida, 2005).
- ²⁹G. J. Simitses and D. H. Hodges, *Fundamentals of Structural Stability* (Elsevier, Burlington, 2006).
- ³⁰W. T. Koiter, *Elastic Stability and Solids and Structures*, edited by A. M. A. Van der Heijden (Cambridge University Press, Cambridge, 2009).
- ³¹N. Challamel, S. A. Meftah, and F. Bernard, *Mech. Res. Commun.* **37**, 472 (2010).
- ³²F. Y. Cheng and C. P. Pantelides, *J. Struct. Eng.* **114**, 1152 (1988).
- ³³C. M. Wang, Y. Xiang, and S. Kitipornchai, *J. Eng. Mech., ASCE* **117**, 1973 (1991).
- ³⁴C. M. Wang, Y. Y. Zhang, S. S. Ramesh, and S. Kitipornchai, *J. Phys. D: Appl. Phys.* **39**, 3904 (2006).
- ³⁵J. N. Reddy and S. D. Pang, *J. Appl. Phys.* **103**, 023511 (2008).