Informative Starting Points and Option Prices

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Abstract

A common reasoning process is to rely on an informative starting point which is somewhat incorrect and then attempt to adjust it appropriately. Evidence suggests that underlying stock volatility is such a starting point, which is scaled-up to estimate call option volatility. I adjust Black-Scholes, Heston, and Bates models for reliance on this starting point. Adjusted Black-Scholes explains implied-volatility skew and other puzzles. Adjusted Heston stochastic volatility model matches the same data better, does so at more plausible values, while generating a steep short term skew. Furthermore, two novel predictions are empirically tested and are strongly supported in the data.

Keywords: Anchoring-and-Adjustment Heuristic, Implied Volatility Skew, Option Pricing Puzzles, Black-Scholes Model, Heston Model, Bates Model.

JEL Classification: G13, G12, G02

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Informative Starting Points and Option Prices

How many days does Mars take to go around the Sun? When did George Washington become the first president of America? What is the freezing temperature of vodka? When faced with these questions, people typically reason as follows: Earth takes 365 days to go around the Sun. Mars is farther from the Sun than Earth is, so it must take longer. Consequently, they start from 365 days and add to it. USA became a country in 1776, and it might have taken a few years to elect the first president, so they start from 1776 and add to it. Vodka is still liquid when water freezes, so they start from 0 Celsius and subtract from it.

The above examples illustrate a very common reasoning process, which is to rely on an informative starting point and then attempt to adjust it properly (Epley and Gilovich 2006, 2001). In fact, this way of reasoning may be the optimal response of a Bayesian decision-maker facing finite computational resources (Lieder, Griffiths, and Goodman 2013).

A robust finding from psychology and economics literature is that such adjustments tend to be insufficient leaving the final answer biased towards the starting value (see Furnham and Boo 2011 for a review of a large literature). The adjustments are insufficient because people tend to stop adjusting once a plausible value is reached (Epley and Gilovich 2006, 2001). “People may spontaneously anchor on information that readily comes to mind and adjust their response in a direction that seems appropriate, using what Tversky and Kahneman (1974) called the anchoring and adjustment heuristic. Although this heuristic is often helpful, the adjustments tend to be insufficient, leaving people’s final estimates biased towards the initial anchor value.” (Epley and Gilovich (2001) page. 1).

A call option is a leveraged position in the underlying stock. It follows that its volatility must be a scaled-up version of underlying stock volatility. The correct scaling-up factor differs from model to model. For example, in the Black-Scholes model, the correct scaling-up factor is equal to the option price elasticity w.r.t the underlying stock price. In the Heston stochastic volatility model, the correct scaling-up factor depends on option Greeks.

Call option and underlying stock payoffs are joined at the hip and move together in the same direction. This strong co-movement in the same direction makes underlying stock volatility a natural starting point for call option volatility. Given the fact that call option volatility is equal to a scaled-up version of underlying stock volatility, I study the implications of using underlying stock volatility as an informative starting point, and scaling-up, to estimate call option volatility.
As discussed earlier, adjustments to starting points tend to be insufficient. Evidence suggesting that investors insufficiently scale-up underlying stock volatility to estimate call option volatility is summarized in section 1. I adjust Black-Scholes (1973), Heston (1993), and Bates (1996) models for such insufficient scaling-up. The adjustment brings only one change in these models: The risk-free rate, \( r \), is replaced with \( r + \delta \cdot (1 - m) \) where \( \delta \) is the risk-premium on the underlying stock, and \( m \) is the fraction of correct scaling-up factor applied to underlying stock volatility to estimate call option volatility. With correct scaling-up, that is, with \( m = 1 \), the adjusted models revert back to their original counterparts. So, the adjusted models are almost as simple and as easy to calibrate as the original models.

I show that the adjusted models substantially outperform their original counterparts: 1) A comparison of adjusted Black-Scholes with the original Black-Scholes shows that certain data patterns such as the implied volatility skew and leverage adjusted returns, which are puzzles in the original Black-Scholes\(^2\), are explained in the adjusted Black-Scholes. 2) A comparison of the Heston model with the adjusted Heston model shows that the adjusted model: a) matches the same data better, b) does so at more plausible parameter values, and c) easily generates a steep short-term skew. The inability to generate a steep short-term skew has been the Achilles heel of the Heston model (Mikhailov and Nogel 2003). By easily generating a steep short-term skew, the adjustment helps in alleviating this problem.

Furthermore, two novel predictions of the new approach are empirically tested with nearly 26 years of options data. The predictions are strongly supported in the data.

This paper is line with the argument in Shefrin (2010) that there is a need to enrich the mathematical structure of option pricing theory by incorporating more realistic and psychologically relevant assumptions in the models. By imputing approximate thinking to investors, this paper is also broadly consistent with the arguments in Derman (2012). This paper is also consistent with Chance (2003) who argues that we need to look beyond distributional assumptions to some economic mechanism in order to enrich option pricing theory.

This paper adds to the nascent literature that studies the implications of reliance on informative starting points for financial markets. Siddiqi (2016a) adjusts the capital asset pricing model (CAPM) for reliance on informative starting points and finds that explanatory power of

\(^2\) See Coval and Shumway (2001), Whaley (2002), Constantinides et al (2013), and Bondarenko (2014) for a sample of articles that discuss these empirical puzzles.
CAPM goes up substantially with this adjustment. Siddiqi (2016b) adjusts the consumption CAPM for such reliance and puts forward a unified explanation for prominent puzzles.

This paper is organized as follows. Section 1 summarizes evidence indicating that investors insufficiently scale-up underlying stock volatility to estimate call option volatility. Section 2 provides a simple numerical example of insufficient adjustment and motivates key results. Section 3 derives some general results which are directly useful in adjusting option pricing models. Section 4 presents the adjusted formulas. Section 5 compares the adjusted Black-Scholes with the original Black-Scholes and shows that the adjusted model internalizes puzzling data patterns. Section 6 compares the adjusted Heston model with the original Heston model and shows that the adjusted model matches the same data better, does so at more plausible values, and easily generates a steep short-term skew. Section 7 presents and tests two novel predictions. Section 8 concludes.

1. Evidence of Insufficient Adjustment

In this section, I summarize empirical, experimental, and anecdotal evidence consistent with the notion that investors insufficiently scale-up underlying stock volatility to estimate call option volatility.

Even though volatility itself is not directly observable, the implications of underestimating volatility are directly observable. According to asset pricing theory, all assets must satisfy the Euler equation (Cochrane 2005). For the underlying stock, and a call option defined on the stock, we have:

\[ 1 = E[SDF \cdot R_S] \]
\[ 1 = E[SDF \cdot R_C] \]

where \( SDF = \frac{\beta u'(c_{t+1})}{u'(c_t)} \). That is, \( SDF \) is the intertemporal marginal rate of substitution. Returns on the underlying stock, and the call option are \( R_S \) and \( R_C \) respectively. The above can also be written as:

\[ E[R_S] - R_F = -\rho_S \cdot \frac{\sigma(SDF)}{E[SDF]} \cdot \sigma(R_S) \]
\[ E[R_C] - R_F = -\rho_C \cdot \frac{\sigma(SDF)}{E[SDF]} \cdot \sigma(R_C) \]

where \( \rho_S \) and \( \rho_C \) are correlations of stock and call returns with the \( SDF \) respectively.

Call option volatility must be a scaled-up version of underlying stock volatility. That is, \( \sigma(R_C) = \sigma(R_S)(1 + A) \) where \( A > 0 \). Substituting \( \rho_S \approx \rho_C \), and \( \sigma(R_C) = \sigma(R_S)(1 + A) \) in the second equation above and substituting from the first equation leads to the following:

\[ E[R_C] = E[R_S] + (E[R_S] - R_F) \cdot A \]  \( (1.1) \)

From (1.1), one can directly see that lower \( A \) is associated with lower call returns. If the scaling-up factor is under-estimated, that is, if \( A \) is perceived to be smaller than the correct value \( (\bar{A}) \), then it follows that the return demanded for holding a call option must be less than the theoretically correct return. In that case, observed call returns must be lower than what the systematic risk suggests. Coval and Shumway (2001) find that, empirically, call returns have been much smaller than what the systematic risk suggests.

As put option volatility follows deductively from underlying stock and corresponding call option volatility, expected return on a put option can be written, for positive \( a \) and \( b \), as (see appendix D):

\[ E[R_P] = R_F - \delta[a - b(1 + A)] \]  \( (1.2) \)

where \( \delta = (E[R_S] - R_F) \), and \( a > b(1 + A) \).

If perceived \( A < \bar{A} \), then it follows that observed put option returns must be smaller (more negative) than what the systematic risk suggests. Indeed, this is what Bondarenko (2014) finds. Hence, empirical evidence on both the call and put returns is consistent with insufficient adjustment. Even though one possibility is to assume that systematic risk is not correctly measured (there are missing risk factors that somehow lower option returns)\(^3\), insufficient adjustment is a much simpler explanation. At the very least, insufficient adjustment must be considered a plausible explanation for low returns.

Controlled laboratory experiments are useful in isolating the effects of insufficient adjustment in (1.1). It is straightforward to design an experiment in which the values of \( E[R_S] \),

\(^3\) Commenting on this, Bates (2003) writes, “To blithely attribute the divergence between objective and risk-neutral measures to the free ‘risk-premium’ parameters within an affine model is to abdicate once responsibility as a financial economist.”
$R_F$ and the correct value of $A$, or $\bar{A}$, are set by experimental parameters. Furthermore, the simplest cases (binomial and trinomial) can be studied where it is difficult to see a role for unknown risk factors. If observed call returns are smaller than theoretically correct returns in such experiments, then this would be strongly suggestive of perceived $A < \bar{A}$. Siddiqi (2012) (by building on the earlier work in Siddiqi (2011) and Rockenbach (2004)) conducts a series of laboratory experiments and finds that observed call returns tend to be substantially smaller than theoretically correct returns. Once again, this is strongly suggestive of perceived $A < \bar{A}$.

Evidence of such risk-underestimation can be seen in the behavior of professional traders who typically argue that a call option is a good proxy for the underlying stock, and frequently advise clients to replace the underlying stocks in their portfolios with call options.\(^4\)\(^5\) Replacing stocks with calls increases portfolio volatility substantially, so call options are not good proxies for the underlying stocks unless one underestimates this additional volatility. It is generally correct to say that replacing stocks with calls increases the overall risk. However, professional advice on this strategy often touts its risk-reducing advantages.\(^6\) Hence, the puzzling popularity of the stock-replacement-with-call-option strategy is in line with professional analysts underestimating call option risk relative to the underlying stock risk.

### 2. Implications of Insufficient Adjustment: A Numerical Example

Imagine that there are 4 types of assets in the market with payoffs shown in Table 1. The assets are a risk-free bond, a stock, a call option on the stock with a strike of 100, and a put option on the stock also with a strike of 100. There are two states of nature labeled Green and Blue that have equal probability of occurrence. The risk-free asset pays 100 in each state, the stock price is 200 in the Green state, and 50 in the Blue state.

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\(^4\) A few examples of experienced professionals stating this are:  
http://www.optionstrading.org/strategies/other/stock-replacement/  

\(^5\) Jim Cramer, the host of popular US finance television program “Mad Money” (CNBC) has contributed to making this strategy widely known among general public.

\(^6\)  
Table 1

<table>
<thead>
<tr>
<th></th>
<th>Bond</th>
<th>Stock</th>
<th>Call</th>
<th>Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>Green State</td>
<td>100</td>
<td>200</td>
<td>100</td>
<td>0</td>
</tr>
<tr>
<td>Blue State</td>
<td>100</td>
<td>50</td>
<td>0</td>
<td>50</td>
</tr>
</tbody>
</table>

What are the equilibrium implications of using underlying stock volatility as an informative starting point for call option volatility? We consider two cases: 1) The rational expectations case which corresponds to correct adjustment to the starting point. 2) Insufficient adjustment case.

2.1 The Rational Expectations Case

Suppose the market is described by a representative agent who faces the following decision problem:

$$\max u(C_0) + \beta E[u(C_1)]$$

subject to

$$C_0 = e_0 - S \cdot n_s - C \cdot n_c - P \cdot n_p - P_F \cdot n_F$$

$$\tilde{C}_1 = e_1 + \tilde{X}_s \cdot n_s + \tilde{X}_c \cdot n_c + \tilde{X}_p \cdot n_p + X_F \cdot n_F$$

where $C_0$ and $C_1$ are current and next period consumption, $e_0$ and $e_1$ are endowments, $S, C, P,$ and $P_F$ denote the prices of stock, call option, put option, and the risk-free asset, and $\tilde{X}_s, \tilde{X}_c, \tilde{X}_p$ and $X_F$ are their corresponding payoffs. The number of units of each asset type is denoted by $n_s, n_c, n_p,$ and $n_F$ with the first letter of the asset type in the subscript (letter $F$ is used for the risk-free asset).

The first order conditions are:

$$1 = E[SDF] \cdot E[R_s] + \rho_s \cdot \sigma[SDF] \cdot \sigma(R_s)$$

$$1 = E[SDF] \cdot E[R_c] + \rho_c \cdot \sigma[SDF] \cdot \sigma(R_c)$$

$$1 = E[SDF] \cdot E[R_p] + \rho_p \cdot \sigma[SDF] \cdot \sigma(R_p)$$

$$1 = E[SDF] \cdot R_F$$
where \( E[\cdot] \) and \( \sigma[\cdot] \) are the expectation and standard deviation operators respectively, SDF is the stochastic discount factor or the inter-temporal marginal rate of substitution of the representative investor \( SDF = \frac{\beta u'(c_t)}{u'(c_0)} \), and \( \rho \) denotes the correlation of the asset with the SDF.

One can simplify the first order conditions further by realizing that \( \rho_c = \rho_s, \rho_p = -\rho_s \), and \( \sigma(X_s) = \sigma(X_c) + \sigma(X_p) \). The last condition captures the fact that stock payoff volatility must either show up in call payoff volatility or the corresponding put payoff volatility by construction. One can see this in Table 1 as well where the stock payoff volatility is 75, the call payoff volatility is 50, and the put payoff volatility is 25. It follows that \( \sigma(R_p) = \frac{S}{p} \cdot \sigma(R_s) - \frac{C}{p} \cdot \sigma(R_c) \). It is easy to see that with payoffs in Table 1, \( \rho_s = -1 \).

The first order conditions can be written as:

\[
1 = E[SDF] \cdot \frac{125}{S} - \sigma[SDF] \cdot \frac{75}{S} \\
1 = E[SDF] \cdot \frac{50}{C} - \sigma[SDF] \cdot \frac{50}{C} \\
1 = E[SDF] \cdot \frac{25}{P} + \sigma[SDF] \cdot \left( \frac{S}{p} \cdot \frac{75}{S} - \frac{C}{p} \cdot \frac{50}{C} \right) \\
1 = E[SDF] \cdot \frac{100}{P_F}
\]

Options must be in zero net supply. Assume that the utility function is \( lnC, \beta = 1, e_0 = e_1 = 500 \), and the representative agent holds one unit of stock and the risk-free asset to clear the market. The above first order conditions can be used to infer the following equilibrium prices: \( P_F = 53.51, S = 62.73, C = 23.99, \text{and } P = 14.76 \).

The key points are:

1) Put-Call parity is satisfied.

2) The binomial model gives the same prices as the equilibrium approach. Call option is replicated by +2/3 of stock and -1/3 of bond. The replication cost is 23.99, which the price of the call option obtained via the equilibrium approach. Put option is replicated by +2/3 of bond
and -1/3 of stock. The replication cost is 14.76 which is equal to put option price obtained via the equilibrium approach.

3) Both (1.1) and (1.2) are satisfied with the correct value of $A$. The correct value of $A = \Omega - 1$. Here, $\Omega = \frac{s}{c} \cdot x$ where $x$ is the number of shares of the stock in the replicating portfolio that mimics the call option. So, in this case, $\Omega = 1.74$ which means the correct value of $A$ or $\tilde{A} = 0.74$.

### 2.2 The Insufficient Adjustment Case

Next, I introduce insufficient adjustment in the picture. The representative investor uses the volatility of the underlying stock as a starting point, which is scaled-up to estimate volatility of call option with the scaling-up factor allowed to be less than the correct value. The first order conditions with insufficient adjustment can be written as:

\[
1 = E[SDF] \cdot \frac{125}{S} - \sigma[SDF] \cdot \frac{75}{S}
\]

\[
1 = E[SDF] \cdot \frac{50}{C} - \sigma[SDF] \cdot \frac{75}{S} \cdot (1 + A)
\]

\[
1 = E[SDF] \cdot \frac{25}{P} + \sigma[SDF] \cdot \left\{ \frac{S}{P} \cdot \frac{75}{S} - \frac{C}{P} \cdot \frac{75}{S} \cdot (1 + A) \right\}
\]

\[
1 = E[SDF] \cdot \frac{100}{P_F}
\]

The only thing different now is the volatility perception of the call option. Instead of $\frac{50}{C}$, the volatility is estimated as $\frac{75}{S} \cdot (1 + A)$. If $A$ takes the correct value of 0.74, we are back to the prices calculated earlier. However, if there is insufficient scaling-up, the results are different. Using the same parameter values as in section 2.1, the equilibrium prices corresponding to $A = 0$ and $A = 0.5$ are shown in Table 2.

The key points are:

1) Put-Call parity continues to hold irrespective of the assumed value of $A$.

2) Equations (1.1) and (1.2) hold with the corresponding assumed value of $A$. 
Table 2

Underestimating Call Option Volatility

<table>
<thead>
<tr>
<th></th>
<th>A=0</th>
<th>A=0.5</th>
<th>A̅=0.74</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>(Correct value)</td>
</tr>
<tr>
<td>S</td>
<td>62.73</td>
<td>62.73</td>
<td>62.73</td>
</tr>
<tr>
<td>Pₚ</td>
<td>53.51</td>
<td>53.51</td>
<td>53.51</td>
</tr>
<tr>
<td>C</td>
<td>25.09</td>
<td>24.34</td>
<td>23.99</td>
</tr>
<tr>
<td>P</td>
<td>15.87</td>
<td>15.11</td>
<td>14.76</td>
</tr>
<tr>
<td>Put-Call Parity</td>
<td>Holds</td>
<td>Holds</td>
<td>Holds</td>
</tr>
<tr>
<td>Equations (1.1) and (1.2)</td>
<td>Hold</td>
<td>Hold</td>
<td>Hold</td>
</tr>
</tbody>
</table>

3) With insufficient scaling-up of underlying stock volatility to estimate call option volatility, both types of options are overpriced compared to correct scaling-up. With A = 0, both options are overpriced by an amount equal to 1.11. A riskless arbitrage opportunity (sell the overpriced option and buy the replicating portfolio) exists unless there are transaction costs. Allowing for proportional transaction costs, arbitrage is precluded in both options with a transaction costs of slightly over 1%. With A = 0.5, the options are overpriced by 0.35 and transaction cost of less than 0.5% is sufficient to prevent arbitrage.

As the example discussed here is a complete market example (options are replicable by using the underlying stock and the risk-free asset), some transaction costs are necessary to prevent arbitrage. In incomplete markets, underestimating call volatility does not automatically create a riskless arbitrage opportunity. The rational option pricing bounds in incomplete markets are derived in Ritchken (1985) which are extended to incorporate proportional transaction costs in Constantinides and Perrakis (2002). We will see that using underlying stock volatility as an informative starting point and insufficiently scaling-up leads to option prices that lie within these bounds; hence, it is not possible to make arbitrage profits against such investors.
3. Equilibrium Implications of Underestimating Call Option Volatility

In the previous section, implications of underestimating the scaling-up factor are numerically illustrated. Using the underlying stock volatility as an informative starting point for call option volatility and failing to adjust fully makes both types of options more expensive.

In this section, I derive the equilibrium implications in a dynamic setting. Consider an exchange economy characterized by a representative agent who maximizes:

\[ E_0 \left[ \sum_{t=0}^{\infty} \beta^t \cdot u(C_t) \right] \]

where \( \beta \) is the time discount factor, and \( C_t \) denotes consumption at time \( t \). What is left after consumption is invested in a portfolio of risky financial assets, one risk-free asset, and a non-financial asset such as human capital or housing wealth.

The agent’s wealth evolves according to:

\[ W_{t+1} = (W_t - C_t) \left( 1 - \sum_{i=1}^{N} \theta_{it} - \theta_{Ht} \right) R_{Ft} + \sum_{i=1}^{N} \tilde{R}_{it+1} \theta_{it} + \theta_{Ht} \tilde{R}_{Ht+1} \]

where \( W_t \) denotes the agent’s wealth at time \( t \), \( \tilde{R}_{it+1} \) and \( R_{Ft} \) are gross returns from risky asset \( i \) and the risk-free asset respectively. \( \theta_{Ht} \) is the fraction of wealth invested in the non-financial asset that has a return of \( \tilde{R}_{Ht+1} \).

The standard perturbation arguments lead to:

\[ 1 = R_{Ft} E_t \left[ \beta \left( \frac{u'(C_{t+1})}{u'(C_t)} \right) \right] \]

\[ 1 = E_t \left[ \beta \left( \frac{u'(C_{t+1})}{u'(C_t)} \cdot R_{it+1} \right) \right] \]

Writing \( \frac{\tilde{X}_{it+1}}{p_{it}} = R_{it+1}, SDF = \beta \frac{u'(C_{t+1})}{u'(C_t)} \) and using \( E[XY] = E[X]E[Y] + \text{Cov}[X,Y] \), the second equation above is equivalent to:

\[ 1 = E[SDF] \cdot E[R_t] + \rho_t \cdot \sigma[SDF] \cdot \sigma[R_t] \]
where SDF is the stochastic discount factor, $\rho_i$ is the correlation of asset $i$’s returns with the SDF, and $E[\cdot]$ and $\sigma[\cdot]$ are the expectation and standard deviation operators respectively.

$$SDF = \frac{\beta u'(c_{t+1})}{u'(c_t)}.$$ Time subscripts are suppressed for simplicity. Among assets, if there is a stock, a call and a put option on that stock with the same strike price, then the following must hold in equilibrium:

\[
1 = E[SDF] \cdot E[R_s] + \rho_s \cdot \sigma[SDF] \cdot \sigma(R_s)
\]
\[
1 = E[SDF] \cdot E[R_c] + \rho_c \cdot \sigma[SDF] \cdot \sigma(R_c)
\]
\[
1 = E[SDF] \cdot E[R_p] + \rho_p \cdot \sigma[SDF] \cdot \sigma(R_p)
\]
\[
1 = E[SDF] \cdot R_F
\]

The above equations can be simplified further by realizing that $\rho_c \approx \rho_s$, $\rho_p \approx -\rho_s$, and $\sigma(R_p) = a \cdot \sigma(R_s) - b \cdot \sigma(R_c)$, where $a = \frac{s}{p}$ and $b = \frac{c}{p}$. Also, there exists an $A$ such that $\sigma(R_c) = \sigma(R_s)(1 + A)$.

The following simplified equations follow:

\[
1 = E[SDF] \cdot E[R_s] + \rho_s \cdot \sigma[SDF] \cdot \sigma(R_s)
\]
\[
1 = E[SDF] \cdot E[R_c] + \rho_s \cdot \sigma[SDF] \cdot \sigma(R_s)(1 + A)
\]
\[
1 = E[SDF] \cdot E[R_p] - \rho_s \cdot \sigma[SDF] \cdot \sigma(R_s)(a - b(1 + A))
\]
\[
1 = E[SDF] \cdot R_F
\]

It follows that,

\[
E[R_c] = E[R_s] + A \cdot \delta
\] (3.1)
\[
E[R_p] = R_F - \delta[a - b(1 + A)]
\] (3.2)

where $R_F$ is the risk-free rate, $\delta = E[R_s] - R_F$, and $a > b(1 + A)$.
It is clear that the agent’s estimate of $A$ matters for option returns (and prices). Define the correct value of $A$ as $\bar{A} = \frac{\sigma(R_C)}{\sigma(R_S)}$. Insufficient adjustment to the starting point implies that $A = m\bar{A}$ where $0 \leq m < 1$. (3.1) becomes:

$$E[R_C] = E[R_S] + m \left( \frac{\sigma(R_C)}{\sigma(R_S)} - 1 \right) \delta$$

Correct adjustment is a special case corresponding to $m = 1$.

In the Black-Scholes model, $m = 1$ and $\frac{\sigma(R_C)}{\sigma(R_S)} = \Omega$ (which is call price elasticity with respect to underlying stock price).

4. Option Pricing Models Adjusted for Informative Starting Points

The results derived in the previous section (equations 3.3 and 3.2) can be directly used to derive appropriate option pricing models. In this section, three most popular models in practice are adjusted for the impact of investor reliance on the informative starting point of underlying stock volatility while estimating call option volatility. The most basic model, which is the Black-Scholes model, is adjusted first, followed by the popular Heston stochastic volatility model, and Bates stochastic volatility with jumps model.

4.1 Black-Scholes Model with the Informative Starting Point

The continuous-time version of (3.3) is:

$$\frac{1}{\Delta t} E[dc] = \frac{1}{\Delta t} E[dS] \frac{\sigma_{ck}}{\sigma_s} - 1 \cdot \delta$$

Where $C$, and $S$, denote the call price, and the stock price respectively. The subscript $K$ is added to emphasize the dependence of call option volatility on the strike price. $\frac{\sigma_{ck}}{\sigma_s}$ is the ratio of instantaneous call and underlying stock volatilities. If $m = 1$, there is correct adjustment, and if underlying stock dynamics are described by geometric Brownian motion, then the model converges to the Black-Scholes model. If $m < 1$, there is insufficient adjustment, and the two formulas differ.
If the risk-free rate is $r$ and the risk premium on the underlying stock is $\delta$, then, $\frac{1}{dt} \frac{E[dS]}{S} = \mu = r + \delta$.

So, (4.1) may be written as:

$$\frac{1}{dt} \frac{E[dc]}{c} = (r + \delta + m \left( \frac{\sigma_K}{\sigma_S} - 1 \right) \cdot \delta)$$

The underlying stock price follows geometric Brownian motion:

$$dS = \mu S dt + \sigma S dZ$$

where $dZ$ is the standard Brownian process.

Ito’s lemma implies:

$$dC = \left[ \frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2} \right] dt + \sigma S \frac{\partial C}{\partial S} dZ$$

It follows that:

$$E[dC] = \left[ \frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2} \right] dt$$

(4.3)

And,

$$\frac{\sigma (dC)}{\sigma (dS)} = \frac{\sigma_K}{\sigma_S} = \frac{S}{C} \frac{\partial C}{\partial S}$$

(4.3a)

Substituting (4.3) and (4.3a) in (4.2) leads to:

$$\left( r + \delta + m \left( \frac{S}{C} \frac{\partial C}{\partial S} - 1 \right) \cdot \delta \right) C = \frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2}$$

(4.4)

A simple re-arrangement of (4.4) results in:

$$\{(1 - m)\mu + mr\} C = \frac{\partial C}{\partial t} + \{(1 - m)\mu + mr\} S \frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2}$$

$$=> \{r + (1 - m)\delta\} C = \frac{\partial C}{\partial t} + \{r + (1 - m)\delta\} S \frac{\partial C}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2}$$

(4.5)
(4.5) describes the partial differential equation (PDE) that must be satisfied if investors rely on the informative starting point of underlying stock volatility while forming judgments about the call option volatility, and then attempt to scale-up.

To appreciate the difference between the adjusted PDE and the Black-Scholes PDE, consider the case of correct adjustment or \( m = 1 \). With correct adjustment (4.5) becomes:

\[
rc = \frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \frac{\partial^2 C}{\partial S^2} \sigma^2 S^2
\]

(4.6) is the Black-Scholes PDE. That is, with correct adjustment, the adjusted PDE converges to the Black-Scholes PDE as expected.

Constantinides and Perrakis (2002) derive tight upper bound (CP upper bound) on a call option’s price in the presence of proportional transaction costs. Their bound is considered the tightest option pricing bound derived in the literature under general conditions. The CP upper bound is the call price at which the expected return from the call option is equal to the expected return from the underlying stock net of round-trip transaction cost:

\[
\bar{C} = \frac{(1+\theta)S \mathbb{E}[C]}{(1-\theta)\mathbb{E}[S]}
\]

It is easy to see that the call price with insufficient adjustment is always less than the CP upper bound. The investor expects a return from a call option which is at least as large as the expected return from the underlying stock. That is, with insufficient adjustment, \( \frac{\mathbb{E}[C]}{C} \geq \frac{\mathbb{E}[S]}{S} > \frac{(1-\theta)\mathbb{E}[S]}{(1+\theta)S} \). It follows that the maximum price under insufficient adjustment is:

\[
\bar{C}_A < \bar{C} = \frac{(1+\theta)S \cdot \mathbb{E}[C]}{(1-\theta)\mathbb{E}[S]}
\]

Note, that the presence of insufficient adjustment, \( m < 1 \), guarantees that the CP lower bound is also satisfied. The CP lower bound is below the Black-Scholes price. As the price with insufficient adjustment is larger than the Black-Scholes price, it follows that it must be larger than the CP lower bound.

There is a closed form solution to the adjusted PDE given in (4.5). Proposition 1 puts forward the resulting European option pricing formulas.

---

7 See Proposition 1 in Constantinides and Perrakis (2002).
Proposition 1 The formula for the price of a European call is obtained by solving the adjusted PDE. The formula is 

\[ C = \{ SN(d_1^A) - Ke^{-(r+\delta(1-m))(T-t)}N(d_2^A) \} \]

where

\[ d_1^A = \frac{\ln(S/K) + (r+\delta(1-m)+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \quad d_2^A = \frac{\ln(S/K) + (r+\delta(1-m)-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}, \]

with \( 0 \leq m \leq 1 \)

Proof.

See Appendix A.

Corollary 1.1 The formula for the anchoring-adjusted price of a European put option is

\[ Ke^{-r(T-t)}\{ 1 - e^{-\delta(1-m)(T-t)}N(d_2^A) \} - S \left( 1 - N(d_1^A) \right) \]

Proof.

Follows from put-call parity. Equivalently, the formula can also be derived by using a continuous time version of 3.2 and Itô’s lemma for put options.

As proposition 1 shows, the only difference between the Black-Scholes formula and the adjusted Black-Scholes formula is replacement of \( r \) with \( r + (1-m)\delta \) where \( m \) is the fraction of correct scaling-up factor applied to underlying stock volatility to estimate call option volatility, and \( \delta \) is the risk-premium on the underlying stock. If the correct scaling-up factor is applied, that is, if \( m = 1 \), then the adjusted formula converges to the original Black-Scholes formula.

In the next section, the popular stochastic volatility model of Heston (1993) is adjusted for reliance on the starting point of underlying stock volatility.

4.2 Heston Stochastic Volatility Model with the Informative Starting Point

In the Heston model, the stock price and its volatility follow the processes given by:

\[ dS = \mu S dt + \sqrt{V} S dw \]

\[ dV = k(\theta - V) dt + \sigma \sqrt{V} dz \]

\[ E[dwdz] = \rho dt \]
where $V$ is the initial instantaneous variance, $\theta$ is the long run average of $V$, $k$ is the rate at which $V$ moves towards $\theta$, and $\sigma$ is the volatility of volatility parameter. The model reverts to the Black-Scholes model when $\sigma$ and $k$ are set to zero.

From Ito’s Lemma:

$$dC = \left[ \frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + k(\theta - V) \frac{\partial C}{\partial V} + \frac{1}{2} V S^2 \frac{\partial^2 C}{\partial S^2} + \frac{1}{2} \sigma^2 V \frac{\partial^2 C}{\partial V^2} + \rho \sigma S V \frac{\partial^2 C}{\partial S \partial V} \right] dt$$

$$+ \sqrt{V} S \frac{\partial C}{\partial S} dw + \sigma \sqrt{V} \frac{\partial C}{\partial V} dz$$

It follows that:

$$E[dC] = \left[ \frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + k(\theta - V) \frac{\partial C}{\partial V} + \frac{1}{2} V S^2 \frac{\partial^2 C}{\partial S^2} + \frac{1}{2} \sigma^2 V \frac{\partial^2 C}{\partial V^2} + \rho \sigma S V \frac{\partial^2 C}{\partial S \partial V} \right] dt \quad (4.7a)$$

And,

$$\frac{\sigma}{\sigma S} \frac{dC}{dS} = \frac{\sigma_{\text{eq}}}{\sigma_{\text{eq}}} = \frac{S \frac{\partial C}{\partial S} + \sigma \frac{\partial C}{\partial V}}{C \frac{\partial C}{\partial S}} \quad (4.7b)$$

Substituting (4.7a) and (4.7b) in (4.2) and re-arranging leads to:

$$\frac{\partial C}{\partial t} + \{(1 - m)\mu + mr\} S \frac{\partial C}{\partial S} + \{k(\theta - V) - \sigma \delta m\} \frac{\partial C}{\partial V} + \frac{1}{2} V S^2 \frac{\partial^2 C}{\partial S^2} + \frac{1}{2} \sigma^2 V \frac{\partial^2 C}{\partial V^2}$$

$$+ \rho \sigma S V \frac{\partial^2 C}{\partial S \partial V} = \{(1 - m)\mu + mr\} C \quad (4.7c)$$

The above PDE can also be written as:

$$\frac{\partial C}{\partial t} + \{r + (1 - m)\delta\} S \frac{\partial C}{\partial S} + \{k(\theta - V) - \sigma \delta m\} \frac{\partial C}{\partial V} + \frac{1}{2} V S^2 \frac{\partial^2 C}{\partial S^2} + \frac{1}{2} \sigma^2 V \frac{\partial^2 C}{\partial V^2}$$

$$+ \rho \sigma S V \frac{\partial^2 C}{\partial S \partial V} = \{r + (1 - m)\delta\} C \quad (4.7d)$$
(4.7d) is the adjusted PDE corresponding to the Heston model. To compare the adjusted PDE with the original Heston model PDE, assume that the correct scaling-up factor is applied to underlying stock volatility to estimate call option volatility. In other words, substitute \( m = 1 \) in (4.7d).

It follows that:

\[
\frac{\partial C}{\partial t} + rS \frac{\partial C}{\partial S} + \{k(\vartheta - V) - \sigma \delta\} \frac{\partial C}{\partial V} + \frac{1}{2} VS^2 \frac{\partial^2 C}{\partial S^2} + \frac{1}{2} \sigma^2 V \frac{\partial^2 C}{\partial V^2} + \rho \sigma S \frac{\partial^2 C}{\partial S \partial V} = rC
\]

(4.7e) is analogous to the PDE in Heston (1993). Hence, as expected, the adjusted PDE converges to the Heston PDE with correct adjustment.

Proposition 2 provides the solution to 4.7d.

**Proposition 2- Adjusted Heston Model:** The price of a European call option when the spot price dynamics are as in the Heston model is given by:

\[
C = \left\{SP_1 - Ke^{-(r+\delta(1-m))(T-t)}P_2 \right\}
\]

where

\[
\delta = \mu - r
\]

\[
P_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re \left\{ \frac{e^{-i\varphi ln S_t}}{i\varphi f(-i)} \right\} d\varphi
\]

\[
P_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re \left\{ \frac{e^{-i\varphi ln S_t}}{i\varphi} \right\} d\varphi
\]

\[
f_{AH}(\varphi) = e^{A+B+C}
\]

\[
A = i\varphi ln S_t + i\varphi \left( (r + \delta \cdot (1 - m)) \right) (T - t)
\]
\[ B = \frac{\varphi k}{\sigma^2} \left( (k - \rho \sigma i \varphi - d)(T - t) - 2\ln \left( \frac{1 - ge^{-d(T-t)}}{1 - g} \right) \right) \]

\[ C = \frac{V}{\sigma^2} \frac{(k - \rho \sigma i \varphi - d)(1 - e^{-d(T-t)})}{1 - ge^{-d(T-t)}} \]

\[ d = \sqrt{(\rho \sigma i \varphi - k)^2 + \sigma^2 (i \varphi + \varphi^2)} \]

\[ g = \frac{k - \rho \sigma i \varphi - d}{k - \rho \sigma i \varphi + d} \]

**Proof.**

See Appendix B

As proposition 2 shows, the adjusted Heston model differs from the original Heston model in only one way, which is replacement of \( r \) with \( r + (1 - m) \delta \). If the correct scaling-up factor is applied to underlying stock volatility to estimate call option volatility, then the adjusted Heston model converges to the original Heston model.

### 4.3 Bates Model with the Informative Starting Point

Bates model is an extension of Heston model. The dynamics under Bates model are:

\[ dS = (\mu S - \lambda \mu_i)dt + \sqrt{V} Sdw + JSdN \]

\[ dV = k(\theta - V)dt + \sigma \sqrt{V} dz \]

\[ E[dwdz] = \rho dt \]

Time subscripts are suppressed for simplicity. Bates model adds a compound Poisson process with jump intensity \( \lambda \) to the Heston model.
A compound Poisson process is a Poisson process where the jump sizes follow the following distribution:

\[
\log(1 + j) \in N \left( \log(1 + \mu_j) - \frac{\sigma_j^2}{2}, \sigma_j^2 \right)
\]

Using Ito’s lemma for the continuous part and an analogous lemma for the jump part, the adjusted PDE for the price of European call option is:

\[
\frac{\partial C}{\partial t} + \mu S \frac{\partial C}{\partial S} + k(\theta - V) \frac{\partial C}{\partial V} + \frac{1}{2} VS^2 \frac{\partial^2 C}{\partial S^2} + \frac{1}{2} \sigma^2 V \frac{\partial^2 C}{\partial V^2} + \rho \sigma S V \frac{\partial^2 C}{\partial S \partial V} \\
+ \lambda E[C(SY, t) - C(S, t)] - \lambda \mu_j \frac{\partial C}{\partial S} = (r + \delta + m(\theta_K - 1) \cdot \delta) C
\]

(4.8)

where \( C_T = \max\{S - K, 0\} \) and \( \theta_K = \frac{\sigma_K}{\sigma_S} = \frac{S \frac{\partial C}{\partial S}}{c \frac{\partial C}{\partial \theta}} + \frac{\sigma \frac{\partial C}{\partial V}}{c \frac{\partial \theta}{\partial V}} \)

(4.8) can be solved by using Fourier methods as in the case of adjusted Heston model.

Proposition 3 provides the solution.

**Proposition 3-Adjusted Bates Model:** The price of a European call option when the spot price dynamics are as in the Bates model is given by:

\[
C = \left\{ SP_1 - Ke^{-(r + (1-m)\delta)(T-t)} P_2 \right\}
\]

where

\[
\delta = \mu - r
\]

\[
P_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left( \frac{e^{-i\varphi lnK} f(\varphi - i)}{i\varphi f(-i)} \right) d\varphi
\]

\[
P_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \text{Re} \left( \frac{e^{-i\varphi lnK} f(\varphi)}{i\varphi} \right) d\varphi
\]
\[ f(\varphi) = e^{A + B + C} \cdot e^{-\lambda \mu i \varphi (T-t) + \lambda (T-t) \left( (1 + \mu) \frac{1}{2} \text{e}^{2i\varphi} - 1 \right)} \]

\[ A = i \varphi \ln S_t + i \varphi (r + (1 - m) \delta)(T - t) \]

\[ B = \frac{\varphi k}{\sigma^2} \left( (k - \rho \sigma i \varphi - d)(T - t) - 2 \ln \left( \frac{1 - ge^{-d(T-t)}}{1 - g} \right) \right) \]

\[ C = \frac{V}{\sigma^2} \frac{(k - \rho \sigma i \varphi - d)(1 - e^{-d(T-t)})}{1 - ge^{-d(T-t)}} \]

\[ d = \sqrt{(\rho \sigma i \varphi - k)^2 + \sigma^2 (i \varphi + \varphi^2)} \]

\[ g = \frac{k - \rho \sigma i \varphi - d}{k - \rho \sigma i \varphi + d} \]

Proof.

See Appendix C

\[ \blacksquare \]

Just like in the Black-Scholes and the Heston model, the only difference between the adjusted Bates model and the original Bates model is replacement of \( r \) with \( r + (1 - m) \delta \). If the correct scaling-up factor is applied to underlying stock volatility to estimate call option volatility, then the adjusted Heston model converges to the original Heston model.

It is straightforward to verify that, in the incomplete market case of adjusted Heston and adjusted Bates models, option prices are within the rational bounds derived in the literature (Constantinides and Perrakis 2002, Ritchken 1985); hence, arbitrage profits are not possible.
5. Adjusted Black-Scholes vs. Original Black-Scholes

In this section, I compare the adjusted Black-Scholes model with the original Black-Scholes model and show the improvements that the adjusted model brings.

5.1 Adjusted Black-Scholes and the Implied Volatility Skew

If option prices are determined in accordance with the formulas in proposition 1 and the Black-Scholes formula is used to back-out implied volatilities than a skew arises. It is straightforward to see this. There is only one difference between the formulas in proposition 1 and the corresponding Black-Scholes formulas. In the adjusted model, the risk-free rate, \( r \), is replaced by \( r + \delta \cdot (1 - m) \). This difference generates the implied volatility skew whenever \( 0 \leq m < 1 \). It is easy to see that the skew is countercyclical as \( \delta \) is countercyclical. The skew steepens as time to expiry nears. To illustrate this, Figure 1 plots the skew with 3-month, 1-month, and 1-week to maturity (other parameters are: \( S = 100, r = 0, \sigma = 20\%, \delta = 5\%, \text{and } m = 0.2 \)).

![Implied Volatility Skew](image_url)

**Figure 1**
5.2 Adjusted Black-Scholes and Leverage-Adjusted Returns

Leverage adjustment dilutes the beta risk of an option by combining it with a risk-free asset. Leverage adjustment combines each option with a risk-free asset in such a manner that the overall beta risk becomes equal to the beta risk of the underlying stock. The weight of the option in the portfolio is equal to its inverse price elasticity w.r.t the underlying stock’s price.

\[
\beta_{portfolio} = \Omega^{-1} \times \beta_{call} + (1 - \Omega^{-1}) \times \beta_{riskfree}
\]

where \( \Omega = \frac{\partial Call}{\partial Stock} \times \frac{Price}{Call} \) (i.e price elasticity of call w.r.t the underlying stock)

\[
\beta_{call} = \Omega \times \beta_{stock}
\]

\[
\beta_{riskfree} = 0
\]

\[
=> \beta_{portfolio} = \beta_{stock}
\]

When applied to index options, such leverage adjustment, which is aimed at achieving a market beta of one, reduces the variance and skewness and renders the returns close to normal enabling statistical inference.

Constantinides, Jackwerth and Savov (2013) uncover a number of interesting empirical facts regarding leverage adjusted index option returns. Table 3 presents the summary statistics of the leverage adjusted returns. As can be seen, four features stand out in the data: 1) Leverage adjusted call returns are lower than the average index return. 2) Leverage adjusted call returns fall with the ratio of strike to spot. 3) Leverage adjusted put returns are typically higher than the index average return. 4) Leverage adjusted put returns also fall with the ratio of strike to spot.

The above features are sharply inconsistent with the Black-Scholes/Capital Asset Pricing Model prediction that all leverage adjusted returns must be equal to the index average return, and should not vary with the ratio of strike to spot.

Section 5.2.1 considers leverage adjusted call returns under insufficient adjustment and shows that they are consistent with the empirical findings. Section 5.2.1 does the same with leverage adjusted put returns.

5.2.1 Leverage adjusted call returns in adjusted Black-Scholes

Applying leverage adjustment to a call option means creating a portfolio consisting of the call option and a risk-free asset in such a manner that the weight on the option is \( \Omega^{-1} \).
Table 3

Average percentage monthly returns of the leverage adjusted portfolios from April 1986 to January 2012. For comparison, average monthly return on S&P 500 index is 0.86% in the same period.

<table>
<thead>
<tr>
<th></th>
<th>Call</th>
<th>Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>K/S</td>
<td>90%</td>
<td>95%</td>
</tr>
<tr>
<td></td>
<td>100%</td>
<td>105%</td>
</tr>
<tr>
<td></td>
<td>110%</td>
<td>Hi-Lo</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>30 days</td>
<td>0.49</td>
<td>0.24</td>
</tr>
<tr>
<td>(s.e.)</td>
<td>0.24</td>
<td>0.24</td>
</tr>
<tr>
<td>90 days</td>
<td>0.51</td>
<td>0.44</td>
</tr>
<tr>
<td>(s.e.)</td>
<td>0.24</td>
<td>0.24</td>
</tr>
<tr>
<td>90-30</td>
<td>0.03</td>
<td>0.02</td>
</tr>
<tr>
<td>(s.e.)</td>
<td>0.02</td>
<td>0.03</td>
</tr>
</tbody>
</table>

Average monthly returns

<table>
<thead>
<tr>
<th></th>
<th>30 days</th>
<th>(s.e.)</th>
<th>90 days</th>
<th>(s.e.)</th>
<th>90-30</th>
<th>(s.e.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>30 days</td>
<td>0.49</td>
<td>0.24</td>
<td>0.51</td>
<td>0.24</td>
<td>0.03</td>
<td>0.02</td>
</tr>
<tr>
<td>(s.e.)</td>
<td>0.24</td>
<td>0.24</td>
<td>0.24</td>
<td>0.24</td>
<td>0.02</td>
<td>0.03</td>
</tr>
<tr>
<td>90 days</td>
<td>0.51</td>
<td>0.44</td>
<td>0.37</td>
<td>0.44</td>
<td>0.02</td>
<td>0.02</td>
</tr>
<tr>
<td>(s.e.)</td>
<td>0.24</td>
<td>0.24</td>
<td>0.24</td>
<td>0.24</td>
<td>0.02</td>
<td>0.03</td>
</tr>
<tr>
<td>90-30</td>
<td>0.03</td>
<td>0.02</td>
<td>0.16</td>
<td>0.03</td>
<td>0.02</td>
<td>0.02</td>
</tr>
<tr>
<td>(s.e.)</td>
<td>0.02</td>
<td>0.02</td>
<td>0.03</td>
<td>0.03</td>
<td>0.02</td>
<td>0.02</td>
</tr>
</tbody>
</table>

It follows that the leverage adjusted call option return is:

\[
\Omega^{-1} \cdot \frac{1}{\partial t} \cdot E \left[ \frac{dC}{C} \right] + (1 - \Omega^{-1})r
\]  

(5.1)

Substituting from (3.1), (5.1) can be written as:

\[
\Omega^{-1} \left[ r + \delta + m \delta \right] + (1 - \Omega^{-1})r
\]

Realizing that \( \bar{A} = \frac{\sigma(R_c)}{\sigma(R_s)} - 1 = \Omega - 1 \), it follows:

\[
\delta \left( m \cdot (1 - \Omega^{-1}) + \Omega^{-1} \right) + r
\]  

(5.2)

From (5.2) one can see that as the ratio of strike to spot rises, leverage adjusted call return must fall. This is because \( \Omega \) rises with the ratio of strike to spot (\( \Omega^{-1} \) falls).
Note that call price elasticity w.r.t the underlying stock price under the adjusted model is:

$$\Omega_K = \frac{S}{SN(d_4^4) - Ke^{-(r+\delta(1-m))(T-t)}N(d_4^4)} \cdot N(d_4^4)$$  \hspace{1cm} (5.3)

Substituting (5.3) in (5.2) and simplifying leads to:

$$R_{LC} = \mu - \delta \cdot \frac{K}{S} \cdot e^{-(r+\delta(1-m))(T-t)} \cdot \frac{N(d_4^4)}{N(d_4^4)} \cdot (1 - m)$$  \hspace{1cm} (5.4)

Note if $m = 1$, then the leverage call return is equal to the CAPM/Black-Scholes prediction, which is $R_{LC} = \mu$. With insufficient scaling-up, the leverage adjusted call return must be less than the average index return as long as the risk premium is positive. Figure 2 is a representative graph of leverage adjusted call returns with the adjusted Black-Scholes ($r = 2\%$, $\delta = 5\%$, $\sigma = 20\%$, $m = 0.2$). Apart from the empirical features mentioned above, one can also see that as expiry increases, returns rise sharply in out-of-the-money range. One can see the same pattern in Table 3 as well.

![Leverage Adjusted Call Returns](image-url)
5.2.2 Leverage adjusted put returns in adjusted Black-Scholes

The leverage adjusted put option return in the adjusted model can be shown to be as follows:

\[
R_{LP} = \mu + \delta \cdot \frac{K}{S} \cdot e^{-(r + \delta (1-m))(T-t)} \cdot \frac{N(d_2^A)}{1-N(d_2^A)} \cdot (1-m)
\]  

(5.5)

As can be seen from the above equation, the CAPM/Black-Scholes prediction of \( R_{LP} = \mu \) is a special case with \( m = 1 \). That is, the CAPM/Black-Scholes prediction follows if there is correct adjustment. With insufficient adjustment, that is, with \( 0 \leq m < 1 \), leverage adjusted put return must be larger than the underlying return if the underlying risk premium is positive. It is also straightforward to verify that insufficient adjustment implies that \( R_{LP} \) falls as the ratio of strike to spot increases.

Figure 3 is a representative plot of the leverage adjusted put returns for 1, 2, and 3 months to expiry (\( r = 2\%, \delta = 5\%, \sigma = 20\%, m = 0.2 \)). One can also see that returns are falling substantially at lower strikes as expiry increases. One can see the same pattern in Table 3.
5.3 The Profitability of Covered Call Writing in Adjusted Black-Scholes

The profitability of covered call writing is quite puzzling in the Black-Scholes framework. Whaley (2002) shows that BXM (a Buy Write Monthly Index tracking a Covered Call on S&P 500) has significantly lower volatility when compared with the index; however, it offers nearly the same return as the index. In the Black Scholes framework, the covered call strategy is expected to have lower risk as well as lower return when compared with buying the index only. See Black (1975).

The covered call strategy ($S$ denotes stock, $C$ denotes call) is given by:

$$ V = S - C $$

In the adjusted Black-Scholes, this is equal to:

$$ V = S - \left\{ SN(d_1^A) - Ke^{-(r+\delta \cdot (1-m)) (T-t)} N(d_2^A) \right\} $$

$$ => V = (1 - N(d_1^A))S + N(d_2^A)Ke^{-(r+\delta \cdot (1-m)) (T-t)} $$

The corresponding value under the Black Scholes assumptions is:

$$ V = (1 - N(d_1))S + N(d_2)Ke^{-r(T-t)} $$

A comparison of 5.6 and 5.7 shows that covered call strategy is expected to perform much better in the adjusted Black-Scholes when compared with its expected performance in the original Black-Scholes model. In the adjusted model, covered call strategy creates a portfolio which is equivalent to having a portfolio with a weight of $1 - N(d_1^A)$ on the stock and a weight of $N(d_2^A)$ on a hypothetical risk-free asset with a return of $r + \delta \cdot (1-m)$. The stock has a return of $r + \delta$ plus dividend yield. This implies that, with anchoring, the return from covered call strategy is expected to be comparable to the return from holding the underlying stock only.

The presence of a hypothetical risk free asset in 5.6 implies that the standard deviation of covered call returns is lower than the standard deviation from just holding the underlying stock. Hence, the superior historical performance of covered call strategy is consistent with the adjusted model.

5.4 The Zero-Beta Straddle Performance in Adjusted Black-Scholes

Another empirical puzzle in the Black-Scholes/CAPM framework is that zero beta straddles lose money. Goltz and Lai (2009), Coval and Shumway (2001) and others find that zero beta
straddles earn negative returns on average. This is in sharp contrast with the Black-Scholes/CAPM prediction which says that the zero-beta straddles should earn the risk-free rate. A zero-beta straddle is constructed by taking a long position in corresponding call and put options with weights chosen so as to make the portfolio beta equal to zero:

\[ \theta \cdot \beta_{\text{Call}} + (1 - \theta) \cdot \beta_{\text{Put}} = 0 \]

\[ \Rightarrow \theta = \frac{-\beta_{\text{Put}}}{\beta_{\text{Call}} - \beta_{\text{Put}}} \]

Where \( \beta_{\text{Call}} = N(d_1^A) \cdot \frac{\text{Stock}_{\text{Call}}}{\text{Call}} \cdot \beta_{\text{Stock}} \) and \( \beta_{\text{Put}} = (N(d_1^A) - 1) \cdot \frac{\text{Stock}_{\text{Put}}}{\text{Put}} \cdot \beta_{\text{Stock}} \)

It is straightforward to see that in the adjusted model, where call and put prices are determined in accordance with proposition 1, the zero-beta straddle earns a significantly smaller return than the risk-free rate (with returns being negative for a wide range of realistic parameter values). Intuitively, in the adjusted model, both call and put options are more expensive when compared with Black-Scholes prices. Hence, the returns are smaller, and are typically negative.

6. Adjusted Heston Model vs. Original Heston Model

In this section, the improvement with the Heston model is shown. Heston model is arguably the most popular stochastic volatility model in practice. The adjusted Heston model (proposition 2) has two additional parameters compared with the original Heston model. The two additional parameters are: \( \delta, \text{and } m \).

The forward-looking estimate of the risk-premium on the underlying, \( \delta \), can be obtained from consensus target price forecasts. Professional analysts publish forecasted prices suggested by their analysis known as target price forecasts. These forecasted prices are widely available. Nasdaq and Yahoo Finance publish corresponding average forecasts (known as consensus forecasts) along with stock information.

6.1 Model Calibration: Google Stock

Before calibrating with S&P 500 index options across various maturities (section 6.2), it is useful to consider a single stock so that item-by-item comparison at the level of prices can be visually
made. European call option prices (midpoint of bid and ask quotes) on Google stock on September 8, 2016 are used for this purpose (Yahoo Finance). These options expire on September 30. The spot price on the day was $775.32. The second column in Table 5 presents this data.

The calibration problem is as follows:

\[
Min(\Delta) = \sum_{i=1}^{N} (C_i - C_i^M)^2
\]

subject to \(2k\theta > \sigma^2, -1 \leq \rho \leq 1, k > 0, 0 \leq \theta \leq 1\) (Heston Model)

subject to \(2k\theta > \sigma^2, -1 \leq \rho \leq 1, k > 0, 0 \leq \theta \leq 1, 0 \leq m \leq 1\) (Adjusted Heston Model).

In the Heston model, the set of calibrated parameters is, \(\Delta = \{k, \theta, V, \rho, \sigma\}\), whereas, in the adjusted Heston model, \(\Delta = \{k, \theta, V, \rho, \sigma, m\}\). So, in the adjusted model, one additional parameter, \(m\), needs to be calibrated, and there is one additional constraint, \(0 \leq m \leq 1\).

The adjusted model replaces \(r\) in the Heston model with \(r + \delta \cdot (1 - m)\). As discussed earlier, to estimate \(\delta\), consensus target price forecasts can be used. Almost all analysts were bullish on Google and the 12-month consensus target price forecast for Google on September 8 is $925 (Yahoo Finance). It follows that \(\delta = 0.190056\) as \(r = 0.003\).

Table 4 shows the results. The sum-of-squared-errors (SSE) with the Heston model is 150.4409. The SSE with the adjusted Heston model is only 24.024. Hence, an improvement nearly by a factor of seven is seen in the adjusted model. This substantial improvement in the model fit is accompanied by more plausible parameter values (in particular, \(\sigma\) and \(\rho\) are smaller) as can be seen in Table 4. The calibrated anchoring parameter is: \(m = 0.643039\). In other words, investors roughly go 65% of the way while adjusting away from the informative starting point of underlying volatility, implying that call option volatility is underestimate by 35%.
Table 4
Comparison of Heston Model with the Adjusted Heston Model. European Call Option Prices on Google Stock are Calibrated on Sept. 8, 2016

<table>
<thead>
<tr>
<th></th>
<th>Heston</th>
<th>Adjusted Heston</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>3.29051</td>
<td>1.848362</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.084983</td>
<td>0.042038</td>
</tr>
<tr>
<td>$V$</td>
<td>8.85E-06</td>
<td>5.89E-05</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.747845</td>
<td>0.01</td>
</tr>
<tr>
<td>$\rho$</td>
<td>-0.54684</td>
<td>-0.20102</td>
</tr>
<tr>
<td>$m$</td>
<td>0.747845</td>
<td>0.64303</td>
</tr>
<tr>
<td>$SSE$</td>
<td>150.4409</td>
<td>24.0224</td>
</tr>
</tbody>
</table>

A key problem with any model with stochastic volatility is that the calibrated volatility-of-volatility parameter, $\sigma$, is much larger than what is plausible based on observed time-series (Bakshi et al 1997). The adjustment helps in alleviating this problem by lowering the calibrated value of $\sigma$ substantially.

Another way to see the improvement is by making individual price-by-price comparison. The last two columns in Table 5 compare the predicted Heston price with the corresponding prediction from the adjusted model. As can be seen, for almost all of the options, the adjusted Heston price is closer to the actual price than Heston price.
Table 5
Individual Price-by-Price Comparison of Adjusted Heston with Original Heston Model.
(A much better fit is obtained with the Adjusted Heston Model). European Call Option
Prices on Google Stock are Calibrated on Sept. 8, 2016

<table>
<thead>
<tr>
<th>Strike</th>
<th>Actual Price</th>
<th>Heston Model</th>
<th>Adjusted Heston Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>450</td>
<td>333</td>
<td>325.4899</td>
<td>331.1602</td>
</tr>
<tr>
<td>675</td>
<td>108.5</td>
<td>100.4768</td>
<td>104.5399</td>
</tr>
<tr>
<td>690</td>
<td>88.4</td>
<td>85.42254</td>
<td>89.08372</td>
</tr>
<tr>
<td>700</td>
<td>78.9</td>
<td>75.52896</td>
<td>79.13071</td>
</tr>
<tr>
<td>730</td>
<td>48.55</td>
<td>45.67375</td>
<td>48.72834</td>
</tr>
<tr>
<td>750</td>
<td>26.6</td>
<td>26.49091</td>
<td>28.02397</td>
</tr>
<tr>
<td>755</td>
<td>21.8</td>
<td>21.87551</td>
<td>22.69437</td>
</tr>
<tr>
<td>760</td>
<td>18.2</td>
<td>17.50127</td>
<td>18.19463</td>
</tr>
<tr>
<td>765</td>
<td>13</td>
<td>13.45726</td>
<td>13.37671</td>
</tr>
<tr>
<td>SSE</td>
<td></td>
<td>150.4409</td>
<td>24.0224</td>
</tr>
</tbody>
</table>

Table 6
Average Parameter Values

<table>
<thead>
<tr>
<th>Heston Model</th>
<th>Adjusted Heston Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>$\theta$</td>
</tr>
<tr>
<td>0.58087</td>
<td>0.00589</td>
</tr>
<tr>
<td>1.79155</td>
<td>0.48351</td>
</tr>
</tbody>
</table>
6.2 Model Calibration: S&P 500 index Options

In this section, European call option prices (midpoint of bid and ask quotes) on S&P 500 index are calibrated. Prices were obtained at the close of September 9, 2016 for the following maturities: 21 days, 42 days, 52 days, 98 days, and 112 days (Yahoo Finance). The spot value was 2127.81. These options are among the most liquid and are traded for a large number of strikes (the lowest strike is $100 and the largest strike is $2300 in the dataset). As per standard practice, each maturity is calibrated separately, and the average parameter values across maturities are reported.

Leading wall-street firms regularly publish 12-month target price forecasts for S&P 500 index. There is no consensus direction for the market among these firms with the forecasts ranging from 2000 (BoA-Merrill Lynch and JP Morgan) at the low end to 2300 (Oppenheimer) at the high end, with other firms distributed evenly in-between. Considering this, all target prices between 2000 and 2300 are used with an interval of 50, $\delta$ is inferred at each, with the model calibrated separately at each $\delta$.

Table 6 shows the average parameter values. On average, the adjusted model achieved a substantially better fit with an average sum-of-squared-error of only 78% of the error from the Heston Model. This better fit is obtained at lower values of $\sigma$ and $\rho$. The adjustment parameter is 0.61 indicating that investors go roughly 61% of the correct way while adjusting away from the underlying volatility.

6.3 Steep Short Term Skew

Generating a steep short-term skew is difficult with the Heston model. This difficulty has been termed as its Achilles heel (Mikhailov and Nogel 2003). Adjusting the Heston model for reliance on the informative starting point of underlying volatility and then insufficiently adjusting away from it leads to the adjusted formula presented in Proposition 2. The adjusted formula generates a steep short-term skew even when the unadjusted model skews are almost completely flat.

It is straightforward to see this. The skew is steeper because $r$ is replaced with $r + \delta(1 - m)$. Figure 4 illustrates this. Both the almost flat Heston model skew and steep adjusted Heston model skews are shown (parameters are: $S = 100, r = 0, T - t = 7$ days, $\delta = 5\%, m = 0.2, k = 2, \theta = 0.2, V = 0.2, \sigma = 0.05, \rho = -0.5$).
7. Novel Predictions

Results regarding leverage-adjusted returns (presented in section 5.2) are not specific to the adjusted Black-Scholes model. Equivalent results are obtained with adjusted-Heston and adjusted Bates models as well. In other words, the results regarding leverage-adjusted returns are due to insufficient adjustment and are not due to the distributional assumptions. In section 5.2, the results are shown with geometric Brownian motion. In this section, I show the same results with stochastic volatility as per the adjusted-Heston model.

Recall, that the instantaneous expected return on a call option is given by:

\[
\frac{1}{C} \frac{E'[dC]}{dt} = (r + \delta + m \left( \frac{\sigma_{cK}}{\sigma_s} - 1 \right) \cdot \delta)
\]

where \( \frac{\sigma_{cK}}{\sigma_s} \) is the ratio of instantaneous call option and underlying stock volatilities. With the assumption of geometric Brownian motion, \( \frac{\sigma_{cK}}{\sigma_s} = \Omega \) where \( \Omega \) is call price elasticity with respect
to the underlying stock price. With stochastic volatility, the ratio of instantaneous volatilities is not equal to $\Omega$; however, it remains a good first approximation. Writing $\frac{\sigma_c K}{\sigma_s} \approx \Omega$, the leverage-adjusted expected return on a call option is:

$$
\delta \{ m \cdot (1 - \Omega_K^{-1}) + \Omega_K^{-1} \} + r
$$

In the adjusted-Heston model:

$$
\Omega_K = \frac{S}{(SP_1 - KE^{-(r + \delta(1-m))(T-t)}P_2) \cdot P_1}
$$

It follows that the leverage-adjusted call option return in the adjusted-Heston model is:

$$
R_{LC} = \mu - \delta \cdot \frac{K}{S} \cdot e^{-(r + \delta(1-m))(T-t)} \cdot \frac{P_2}{P_1} \cdot (1 - m)
$$

(7.1)

Similarly, the leverage-adjusted put option return in the adjusted-Heston model is:

$$
R_{LP} = \mu + \delta \cdot \frac{K}{S} \cdot e^{-(r + \delta(1-m))(T-t)} \cdot \frac{P_2}{(1 - P_1)} \cdot (1 - m)
$$

(7.2)

(7.1) and (7.2) are analogous to (5.4) and (5.5) and it is straightforward to see that figures similar to Figure 2 and Figure 3 can be drawn for the adjusted-Heston model as well.

Two novel predictions arising from insufficient adjustment can be seen directly from Figure 2 and Figure 3.

**Prediction 1.** At low strikes ($K < S$), the difference between leverage adjusted put and call returns must fall as the ratio of strike to spot increases at all levels of expiry.

Figure 3 shows a very sharp dip in leverage adjusted put returns at low strikes. The dip is so sharp that it should dominate the difference between put and call returns in the low strike range. At higher strikes, the decline in put and call returns is of the same order of magnitude.

**Prediction 2.** The difference between leverage adjusted put and call returns must fall as expiry increases at least at low strikes.

Figure 3 shows that put returns fall drastically with expiry at low strikes. They rise marginally at higher strikes with expiry. Figure 2 shows that call returns rise with expiry throughout and

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8 Technical proofs are available from the author upon request.
relatively more so at higher strikes. It follows that the difference between put and call returns should fall with expiry at least at low strikes if not throughout.

Next, I use the dataset developed in Constantinides et al (2013) to test these predictions. Constantinides et al (2013) use Black-Scholes elasticities evaluated at implied volatility for constructing leverage adjusted returns. As the elasticities adjusted for anchoring are close to Black-Scholes elasticities evaluated at implied volatility, the dataset can be used to test the predictions. The dataset used in this paper is available at http://www.wiwi.unikonstanz.de/fileadmin/wiwi/jackwerth/Working_Paper/Version325_Return_Data.txt

The construction of this dataset is described in detail in Constantinides et al (2013). It is almost 26 years of monthly data on leverage adjusted S&P-500 index option returns ranging from April 1986 to January 2012.

7.1. Empirical findings regarding prediction 1

Wilcoxon signed-rank-test is used as it allows for a direct observation by observation comparison of the two time-series. The following procedure is adopted:

1) The dataset has the following ratios of strikes to spot: 0.9, 0.95, 1.0, 1.05, and 1.10. For each value of strike to spot, the difference between leverage adjusted put and call returns is calculated.

<table>
<thead>
<tr>
<th>Put minus Call Return (Monthly)</th>
<th>Put minus Call Return (Monthly)</th>
<th>Maturity (days)</th>
<th>Wilcoxon Signed Rank Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>Leverage Adjusted</td>
<td>Leverage Adjusted</td>
<td></td>
<td>Null Hypothesis: Series 1=Series 2</td>
</tr>
<tr>
<td>(April 1986 to January 2012)</td>
<td>(April 1986 to January 2012)</td>
<td></td>
<td>P-Value</td>
</tr>
<tr>
<td>Series 1 Strike (%spot)</td>
<td>Series 2 Strike (%spot)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.9</td>
<td>0.95</td>
<td>30</td>
<td>5.6288E-14</td>
</tr>
<tr>
<td>0.95</td>
<td>1</td>
<td>30</td>
<td>2.3314E-14</td>
</tr>
<tr>
<td>1</td>
<td>1.05</td>
<td>30</td>
<td>0.0952E+01</td>
</tr>
<tr>
<td>1.05</td>
<td>1.1</td>
<td>30</td>
<td>2.2371E-06</td>
</tr>
<tr>
<td>0.9</td>
<td>0.95</td>
<td>60</td>
<td>1.3105E-09</td>
</tr>
<tr>
<td>0.95</td>
<td>1</td>
<td>60</td>
<td>0.1025E-08</td>
</tr>
<tr>
<td>1</td>
<td>1.05</td>
<td>90</td>
<td>2.8460E-08</td>
</tr>
<tr>
<td>1.05</td>
<td>1.1</td>
<td>90</td>
<td>0.1025E-08</td>
</tr>
</tbody>
</table>
2) Pair-wise comparisons are made between time series of 0.9 and 0.95, 0.95 and 1.0, 1.0 and 1.05, and 1.05 and 1.10. Such comparisons are made for each level of maturity: 30 days, 60 days, or 90 days.

3) The first time-series in each pair is dubbed series1, and the second time-series in each pair is dubbed series 2. That is, for the pair, 0.9 and 0.95, 0.9 is Series 1, and 0.95 is Series 2.

4) For each pair, if the prediction is true, then Series 1>Series 2. This forms the alternative hypothesis in the Wilcoxon signed rank test, which is tested against the null hypothesis: Series 1 = Series 2

Table 7 shows the results. As can be seen from the table, when call is in-the-money, the difference between leverage adjusted put and call returns falls with strike to spot at all levels of expiry (Series 1 is greater than Series 2). Hence, null hypothesis is rejected, in accordance with prediction of the adjusted model. As expected, the p-values are quite large for out-of-the-money call range, so null cannot be rejected for out-of-the-money call range.

**7.2 Empirical findings regarding prediction 2**

To test prediction 2, the procedure adopted is very similar to the one used for prediction 1:

1) For each level of strike to spot, the following pair-wise comparisons are made: 30 days vs 60 days, 60 days vs 90 days, 30 days vs 90 days.

2) The first time-series in each pair is dubbed Series 1, and the second time-series is labeled Series 2. If prediction 2 is true, then Series 1 > Series 2. This forms the alternate hypothesis against the null: Series 1 = Series 2.

3) Wilcoxon signed rank test is conducted for each pair.

Table 8 shows the results. As can be seen, the null is rejected in favor of the alternate hypothesis throughout. Hence, both the novel predictions are strongly supported in the data.
### Table 8

<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Null Hypothesis: Series1=Series2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Alternate: Series1&gt;Series2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>P-Value</td>
</tr>
<tr>
<td>30</td>
<td>60</td>
<td>0.9</td>
<td></td>
<td>0.0000</td>
</tr>
<tr>
<td>60</td>
<td>90</td>
<td>0.9</td>
<td></td>
<td>0.0000</td>
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<td>90</td>
<td>1.1</td>
<td></td>
<td>0.0020</td>
</tr>
</tbody>
</table>

#### 8. Conclusions

A common reasoning approach is relying on an informative starting point and then attempting to adjust it appropriately. Evidence suggests that underlying stock volatility is such a starting point for estimating call option volatility. The first contribution of this article is to show how to adjust popular option pricing models for investor reliance on the informative starting point of underlying stock volatility. The second contribution is to show that adjusted models outperform the corresponding unadjusted models. In particular, several puzzling patterns are explained by the adjusted models. The third contribution is to show that such adjusted models can be calibrated almost as easily as the unadjusted models to see the improvement that such adjustments bring. Furthermore, two novel predictions that arise from reliance on the informative starting point of underlying stock volatility are empirically tested and found to be strongly supported with nearly 26 years of options data.
The door is now open to adjust a wide class of option pricing models spanning currency, commodity and equity markets, for investor reliance on informative starting points. An immediate practical application is fitting the skew better for the purpose of valuing exotic options. Future research should quantify the improvements in a variety of contexts, and with different datasets.

References


Appendix A

Proof of Proposition 1

The formula is obtained by converting the PDE into heat equation and following the same steps as in the derivation of the Black-Scholes model.

Appendix B

Proof of Proposition 2:

The derivation closely follows the derivation in Heston (1993) with \( r \) replaced with \( r + \delta(1-m) \)
Appendix C

Proof of Proposition 3:

The characteristic function for the sum of two independent random variables is the multiplication of the two characteristic functions. In Bates model, log-return is a sum of two independent random variables; one due to stochastic volatility, and the other one due to jumps. So, all we need to do is to multiply the Heston characteristic function with a characteristic function accounting for jumps. This results in the formula in Proposition 3.

Appendix D

The relationship between underlying stock payoff volatility and corresponding call and put option payoff volatilities is given by:

\[ \sigma(X_s) = \sigma(X_c) + \sigma(X_p) \]

Converting payoffs into returns, it follows that:

\[ \sigma(R_p) = \frac{S}{P} \sigma(R_s) - \frac{C}{P} \sigma(R_c) \]

Substituting \( \sigma(R_c) = \sigma(R_s)(1 + A) \) and simplifying leads to:

\[ \sigma(R_p) = \sigma(R_s) \left[ \frac{S}{P} - \frac{C}{P} (1 + A) \right] \]

Setting \( a = \frac{S}{P} \) and \( b = \frac{C}{P} \) and substituting the above expression into the Euler equation for put options leads to:

\[ E[R_p] - R_F = -\rho_p \cdot \frac{\sigma(SDF)}{E[SDF]} \cdot \sigma(R_s) [a - b(1 + A)] \]

Realizing that \( \rho_p \approx -\rho_s \) and that \( E[R_S] - R_F = -\rho_s \cdot \frac{\sigma(SDF)}{E[SDF]} \cdot \sigma(R_S) \) leads to the desired result.