Bipartite 2-factorisations of complete multipartite graphs

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Abstract

It is shown that if $K$ is any regular complete multipartite graph of even degree, and $F$ is any bipartite 2-factor of $K$, then there exists a factorisation of $K$ into $F$; except that there is no factorisation of $K_{6,6}$ into $F$ when $F$ is the union of two disjoint 6-cycles.

1 Introduction

A spanning subgraph of a graph is called a factor, a $k$-regular factor is called a $k$-factor, and a decomposition into edge-disjoint $k$-factors is called a $k$-factorisation. This paper is concerned with 2-factorisations of complete multipartite graphs in which the 2-factors are all isomorphic to a given 2-factor. We shall refer to this problem as the Oberwolfach Problem for complete multipartite graphs, because it is a natural extension from complete graphs to complete multipartite graphs of the well-known
Oberwolfach Problem, which arose out of a seating arrangement problem posed by Ringel at a graph theory meeting in Oberwolfach in 1967. The Oberwolfach Problem for complete multipartite graphs has been studied previously and we shall discuss known results shortly. The purpose of this paper is to give a complete solution (see Theorem 12) in the case where the given 2-factor is bipartite (equivalently, where the given 2-factor is a disjoint union of cycles of even length).

The complete multipartite graph with $r$ parts of cardinalities $s_1, s_2, \ldots, s_r$ is denoted by $K_{s_1, s_2, \ldots, s_r}$, and the notation $K_{s'}$ is used rather than $K_{s_1, s_2, \ldots, s_r}$ when $s_1 = s_2 = \cdots = s_r = s$. The 2-regular graph consisting of $t$ disjoint cycles of lengths $m_1, m_2, \ldots, m_t$ will be denoted by $[m_1, m_2, \ldots, m_t]$, and exponents may be used to indicate multiple cycles of the same length. For example, $[4, 4, 6, 6, 6, 10]$ may be denoted by $[4^2, 6^3, 10]$. Throughout the paper, the meaning of any notation involving an exponent is as defined in this paragraph.

A 2-factorisation in which each 2-factor is a single cycle is a Hamilton decomposition. Auerbach and Laskar [3] proved in 1976 that a complete multipartite graph has a Hamilton decomposition if and only if it is regular of even degree.

**Theorem 1** ([3]) *A complete multipartite graph has a Hamilton decomposition if and only if it is regular of even degree.*

The complete multipartite graph with $n$ parts each consisting of a single vertex is the complete graph on $n$ vertices which is denoted by $K_n$. The problem of finding a 2-factorisation of $K_n$ in which the 2-factors are isomorphic to a given 2-factor $F$ is the Oberwolfach Problem. The Oberwolfach Problem has been completely settled for infinitely many values of $n$ [8], when $F$ consists of cycles of uniform length [2], and in many other special cases. The known results on the Oberwolfach Problem up to 2007 can be found in the survey [7], and several new results appearing after [7] was published are cited in the introduction of [6].

If $n$ is even, then $K_n$ has odd degree and no 2-factorisation exists. However, if $F$ is any given 2-regular graph on $n$ vertices where $n$ is even, then one may ask instead for
a factorisation of $K_n$ into $\frac{n-2}{2}$ copies of $F$ and a 1-factor. The Oberwolfach Problem is now usually considered to include this problem, and solutions are equivalent to 2-factorisations of the complete multipartite graph with $\frac{n}{2}$ parts of cardinality 2. The status of the problem is similar to that of the Oberwolfach Problem for $n$ odd (see the survey [7] and the references cited in [6]), with a notable exception being that the problem has been completely settled in all cases where $F$ is bipartite [5, 10]. Of course, $F$ is never bipartite when $n$ is odd.

**Theorem 2** ([5, 10]) If $F$ is a bipartite 2-regular graph of order $2r$, then the complete multipartite graph $K_{2r}$ has a 2-factorisation into $F$.

Piotrowski [12] has completely settled the Oberwolfach Problem for complete bipartite graphs. Obviously, the 2-factors are necessarily bipartite in this problem.

**Theorem 3** ([12]) If $F$ is a bipartite 2-regular graph of order $2n$, then the complete bipartite graph $K_{n,n}$ has a 2-factorisation into $F$ except when $n = 6$ and $F \cong [6,6]$.

The Oberwolfach Problem for complete multipartite graphs has also been completely settled, by Liu [11], for cases where the 2-factors consist of cycles of uniform length.

**Theorem 4** ([11]) The complete multipartite graph $K_{n^r}$, $r \geq 2$, has a 2-factorisation into 2-factors composed of $k$-cycles if and only if $k$ divides $rn$, $(r-1)n$ is even, $k$ is even when $r = 2$, and $(k,r,n)$ is none of $(3,3,2)$, $(3,6,2)$, $(3,3,6)$, $(6,2,6)$.

In Theorem 12, we generalise Theorems 2 and 3, completely settling the Oberwolfach Problem for complete multipartite graphs in the case of bipartite 2-factors.

## 2 Notation and preliminaries

Let $\Gamma$ be a finite group. A Cayley subset of $\Gamma$ is a subset which does not contain the identity and which is closed under taking of inverses. If $S$ is a Cayley subset of $\Gamma$, then
the Cayley graph on $\Gamma$ with connection set $S$, denoted $\text{Cay}(\Gamma, S)$, has the elements of $\Gamma$ as its vertices and there is an edge between vertices $g$ and $h$ if and only if $g = h + s$ for some $s \in S$.

We need the following two results on Hamilton decompositions of Cayley graphs. The first was proved by Bermond et al [4], and the second by Dean [9]. Both results address the open question of whether every connected Cayley graph of even degree on a finite abelian group has a Hamilton decomposition [1].

**Theorem 5 ([4])** Every connected 4-regular Cayley graph on a finite abelian group has a Hamilton decomposition.

**Theorem 6 ([9])** Every 6-regular Cayley graph on a cyclic group which has a generator of the group in its connection set has a Hamilton decomposition.

A Cayley graph on a cyclic group is called a circulant graph and we will be using these, and certain subgraphs of them, frequently. Thus, we introduce the following notation. The length of an edge $\{x, y\}$ in a graph with vertex set $\mathbb{Z}_m$ is defined to be either $x - y$ or $y - x$, whichever is in $\{1, 2, \ldots, \lfloor \frac{m}{2} \rfloor \}$ (calculations in $\mathbb{Z}_m$). When $m$ is even and $s \leq \frac{m-2}{2}$, we call $\{\{x, x+s\} : x = 0, 2, \ldots, m-2\}$ the even edges of length $s$ and we call $\{\{x, x+s\} : x = 1, 3, \ldots, m-1\}$ the odd edges of length $s$. Note that elsewhere in the literature, the term “even (odd) edges” has sometimes been used for edges of even (odd) length.

For any $m \geq 3$ and any $S \subseteq \{1, 2, \ldots, \lfloor \frac{m}{2} \rfloor \}$, we denote by $\langle S \rangle_m$ the graph with vertex set $\mathbb{Z}_m$ and edge set consisting of the edges of length $s$ for each $s \in S$, that is, $\langle S \rangle_m = \text{Cay}(\mathbb{Z}_m, S \cup -S)$. For $m$ even, if we wish to include in our graph only the even edges of length $s$ then we give $s$ the superscript “e”. Similarly, if we wish to include only the odd edges of length $s$ then we give $s$ the superscript “o”. For example, the graph $\langle \{1, 2^e, 5^e\} \rangle_{12}$ is shown in Figure 1.

The wreath product $G \wr H$ of graphs $G$ and $H$ is the graph with vertex set $V(G) \times V(H)$ and edge set given by joining $(g_1, h_1)$ to $(g_2, h_2)$ precisely when $g_1$ is joined to
Figure 1: The graph \( \langle \{1, 2^0, 5^c\} \rangle_{12} \)

g_2 in G or \( g_1 = g_2 \) and \( h_1 \) is joined to \( h_2 \) in \( H \). We will be dealing frequently with the wreath product of a graph \( K \) and the empty graph with vertex set \( \mathbb{Z}_2 \), so we introduce the following special notation for this graph. The graph \( K^{(2)} \) is defined by

\[
V(K^{(2)}) = V(K) \times \mathbb{Z}_2 \quad \text{and} \quad E(K^{(2)}) = \{ \{(x, a), (y, b)\} : \{x, y\} \in E(K), \ a, b \in \mathbb{Z}_2\}. 
\]

It is easy to see that \( \text{Cay}(\Gamma, S)^{(2)} \cong \text{Cay}(\Gamma \times \mathbb{Z}_2, S \times \mathbb{Z}_2) \). If \( F = \{F_1, F_2, \ldots, F_t\} \) is a set of graphs, then we define \( F^{(2)} = \{F_1^{(2)}, F_2^{(2)}, \ldots, F_t^{(2)}\} \). Note that if \( F \) is a factorisation of \( K \), then \( F^{(2)} \) is a factorisation of \( K^{(2)} \).

Häggkvist [10] observed that for any bipartite 2-regular graph \( F \) on \( 2m \) vertices, there is a 2-factorisation of \( C_m^{(2)} \) into two copies of \( F \). The following very useful result, on which many of our constructions depend, is an immediate consequence of Häggkvist’s observation and the fact that \( F^{(2)} \) is a factorisation of \( K^{(2)} \) when \( F \) is a factorisation of \( K \). If \( F \) is a Hamilton decomposition of \( K \), then we obtain a 4-factorisation of \( K^{(2)} \) into copies of \( C_m^{(2)} \) (where \( m \) is the number of vertices in \( K \)), and we then obtain the required 2-factorisation of \( K^{(2)} \) by factorising each copy of \( C_m^{(2)} \) into two copies of the required bipartite 2-regular graph.

**Lemma 7** ([10]) If there is a Hamilton decomposition of \( K \), then for each bipartite 2-regular graph \( F \) of order \( |V(K^{(2)})| \), there is a 2-factorisation of \( K^{(2)} \) into \( F \).
3 Main Result

We begin this section with two results on factorisations of $K_{m^r}$ in cases where $K_{m^r}$ has odd degree. Note that $K_{m^r}$ has odd degree if and only if $m$ is odd and $r$ is even. In Lemma 8 the factorisation is into Hamilton cycles and a 3-factor isomorphic to $\langle\{1,3^e\}\rangle_{rm}$, and in Lemma 9 the factorisation is into Hamilton cycles and a 5-factor isomorphic to $\langle\{1,2,3^e\}\rangle_{rm}$. These factorisations are used in the proof of Theorem 12.

**Lemma 8** For each even $r \geq 4$ and each odd $m \geq 1$, except $(r,m) = (4,1)$, there is a factorisation of $K_{m^r}$ into $(r-1)m^r - 3$ Hamilton cycles and a copy of $\langle\{1,3^e\}\rangle_{rm}$.

**Proof** First observe that $K_{m^r} \cong \langle\{1,2,\ldots,\frac{rm}{2}\}\rangle \setminus \{r,2r,\ldots,\frac{m-1}{2}r\}$, where $t = (rm - m - 3)/2$. The cases $rm \equiv 0 \pmod{4}$ and $rm \equiv 2 \pmod{4}$ are dealt with separately. For $rm \equiv 2 \pmod{4}$ it is easy to verify that the mapping $\psi : \mathbb{Z}_{rm} \mapsto \mathbb{Z}_{rm}$ given by

$$\psi(x) = \begin{cases} \frac{x}{2} & \text{if } x \equiv 0 \pmod{4} \\ \frac{rm}{2} + \lfloor \frac{x}{2} \rfloor & \text{if } x \equiv 1,2 \pmod{4} \\ \frac{x-1}{2} & \text{if } x \equiv 3 \pmod{4} \end{cases}$$

is an isomorphism from $\langle\{1,3^e\}\rangle_{rm}$ to $\langle\{1,\frac{rm}{2}\}\rangle_{rm}$. So in the case $rm \equiv 2 \pmod{4}$ it is sufficient to show that $\langle\{2,3,\ldots,\frac{rm}{2} - 1\}\rangle \setminus \{r,2r,\ldots,\frac{m-1}{2}r\}_m$ has a Hamilton decomposition.

Consider the sequence $S = s_1, s_2, \ldots, s_t$ (where $t = (rm - m - 3)/2$) whose terms are the elements of

$$\{2,3,\ldots,\frac{rm}{2} - 1\} \setminus \{r,2r,\ldots,\frac{m-1}{2}r\}$$

arranged in ascending order. Note that since $r$ is even, consecutive terms in $S$ are relatively prime. Thus, if $a$ and $b$ are consecutive terms in $S$, then $\langle\{a,b\}\rangle_{rm}$ is connected and thus has a Hamilton decomposition by Theorem 5. Also, since we are in the case $rm \equiv 2 \pmod{4}$, we have $s_{t-1} = \frac{rm}{2} - 2$ and $\gcd(\frac{rm}{2} - 2, rm) = 1$. Thus, $\langle\{s_{t-2},s_{t-1},s_t\}\rangle_{rm}$ has a Hamilton decomposition by Theorem 6.
In view of the arguments in the preceding paragraph, we can obtain the required Hamilton decomposition of $\langle\{2, 3, \ldots, \frac{rm}{2} - 1\}\setminus\{r, 2r, \ldots, \frac{m-1}{2}r\}\rangle_m$ by factoring it into Hamilton decomposable 4-regular graphs of the form $\langle\{a, b\}\rangle_{rm}$ where $a$ and $b$ are consecutive terms of $S$, and, in the case where the number of terms of $S$ is odd, the Hamilton decomposable 6-regular graph $\langle\{s_{t-2}, s_{t-1}, s_t\}\rangle_{rm}$.

Now consider the case $rm \equiv 0 \pmod{4}$. It is easy to see that $\langle\{2, 3^\circ, \frac{rm}{2}\}\rangle_{rm} \cong \text{Cay}(\mathbb{Z}_{\frac{rn}{2}} \times \mathbb{Z}_2, \{(1,0), (\frac{rm}{2}, 0), (0,1)\})$, and that this graph is connected. It follows that $\langle\{2, 3^\circ, \frac{rm}{2}\}\rangle_{rm}$ has a Hamilton decomposition by Theorem 5. Thus, it is sufficient to show that $\langle\{4, 5, \ldots, \frac{rm}{2} - 1\}\setminus\{r, 2r, \ldots, \frac{m-1}{2}r\}\rangle_m$ has a Hamilton decomposition. Redefine $S = s_1, s_2, \ldots, s_t$ to be the sequence whose terms are the elements of $\{4, 5, \ldots, \frac{rm}{2} - 1\}\setminus\{r, 2r, \ldots, \frac{m-1}{2}r\}$ arranged in ascending order (so $t$ is now $(rm - m - 7)/2$). As before, consecutive terms in $S$ are relatively prime.

Since we are in the case $rm \equiv 0 \pmod{4}$, we have $\gcd(\frac{rm}{2} - 1, rm) = 1$, which means that $\langle\{s_{t-2}, s_{t-1}, s_t\}\rangle_{rm}$ has a Hamilton decomposition by Theorem 6. We can thus obtain the required Hamilton decomposition of $\langle\{4, 5, \ldots, \frac{rm}{2} - 1\}\setminus\{r, 2r, \ldots, \frac{m-1}{2}r\}\rangle_m$ by factoring it into Hamilton decomposable 4-regular graphs of the form $\langle\{a, b\}\rangle_{rm}$ where $a$ and $b$ are consecutive terms of $S$, and, in the case where the number of terms of $S$ is odd, the Hamilton decomposable 6-regular graph $\langle\{s_{t-2}, s_{t-1}, s_t\}\rangle_{rm}$. □

**Lemma 9** For each even $r \geq 4$ and each odd $m \geq 3$ such that $rm \equiv 8 \pmod{12}$, there is a factorisation of $K_{mr}$ into $\frac{(r-1)m-5}{2}$ Hamilton cycles and a copy of $\langle\{1, 2, 3^\circ\}\rangle_{rm}$.

**Proof** Since $K_{mr} \cong \langle\{1, 2, \ldots, \frac{rm}{2}\}\setminus\{r, 2r, \ldots, \frac{m-1}{2}r\}\rangle_{rm}$, it is sufficient to show that there is a Hamilton decomposition of $\langle\{3^\circ, 4, 5, \ldots, \frac{rm}{2}\}\setminus\{r, 2r, \ldots, \frac{m-1}{2}r\}\rangle_{rm}$. Note that neither 6 nor $\frac{rm}{2}$ is in $\{r, 2r, \ldots, \frac{m-1}{2}r\}$. Now, it is easy to see that

$$\langle\{3^\circ, 6, \frac{rm}{2}\}\rangle_{rm} \cong \text{Cay}(\mathbb{Z}_{\frac{rn}{2}} \times \mathbb{Z}_2, \{(3,0), (\frac{rm}{4}, 0), (0,1)\})$$

and hence that $\langle\{3^\circ, 6, \frac{rm}{2}\}\rangle_{rm}$ is a connected 4-regular Cayley graph (connectedness follows from $\gcd(3, \frac{rm}{2}) = 1$). Thus, $\langle\{3^\circ, 6, \frac{rm}{2}\}\rangle_{rm}$ has a Hamilton decomposition by
Theorem 5, and it is sufficient to show that \( \langle \{4, 5, \ldots, \frac{rm}{2} - 1\} \setminus \{6, r, 2r, \ldots, \frac{m-1}{2}r\}\rangle_{rm} \) has a Hamilton decomposition.

Consider the sequence \( S = s_1, s_2, \ldots, s_t \) (where \( t = \frac{rm - m - 9}{2} \)) whose terms are the elements of

\[
\{4, 5, \ldots, \frac{rm}{2} - 1\} \setminus \{6, r, 2r, \ldots, \frac{m-1}{2}r\}
\]

arranged in ascending order. Note that since \( r \) is even, consecutive terms in \( S \) are relatively prime. Thus, if \( a \) and \( b \) are consecutive terms in \( S \), then \( \langle \{a, b\}\rangle_{rm} \) is connected and thus has a Hamilton decomposition by Theorem 5. Also, gcd\((s_t, rm) = 1 \) (since \( s_t = \frac{rm}{2} - 1 \) is odd), and so \( \langle \{s_t\}\rangle_{rm} \) is an \( rm \)-cycle.

In view of the preceding paragraph, we can obtain the required Hamilton decomposition of \( \langle \{4, 5, \ldots, \frac{rm}{2} - 1\} \setminus \{6, r, 2r, \ldots, \frac{m-1}{2}r\}\rangle_{rm} \) by factoring it into Hamilton decomposable 4-regular graphs of the form \( \langle \{a, b\}\rangle_{rm} \) where \( a \) and \( b \) are consecutive terms of \( S \), and, in the case where the number of terms of \( S \) is odd, the cycle \( \langle \{s_t\}\rangle_{rm} \).

\[ \square \]

We also need the following result from [6].

**Lemma 10** ([6]) Let \( n \equiv 0 \pmod{4} \) with \( n \geq 12 \). For each bipartite 2-regular graph \( F \) of order \( n \), there is a factorisation of \( \langle \{1, 3^e\}\rangle^{(2)}_{n/2} \) into three copies of \( F \); except possibly when \( F \in \{[6^r], [4, 6^r] : r \equiv 2 \pmod{4}\} \).

In the proof of Theorem 12, an alternate approach is required when \( F \) is one of the possible exceptions in Lemma 10. Cases where \( F \) is of the form \([6^r]\) are covered by Theorem 4, and the following result is used together with Lemma 9 to deal with cases where \( F \) is of the form \([4, 6^r]\).

**Lemma 11** For each \( k \geq 1 \), there is a factorisation of \( \langle \{1, 2, 3^e\}\rangle^{(2)}_{12k+8} \) into five copies of \([4, 6^{4k+2}]\).

**Proof** For any subgraph \( F \) of \( \langle \{1, 2, 3^e\}\rangle^{(2)}_{12k+8} \) and any \( t \in \{0, 2, \ldots, 12k + 6\} \), let \( F + t \) denote the subgraph of \( \langle \{1, 2, 3^e\}\rangle^{(2)}_{12k+8} \) obtained by applying the permutation...
(x, i) \mapsto (x + t, i). That is, V(F + t) = \{(x + t, i) : (x, i) \in V(F)\} and E(F + t) = \{(x + t, i)(y + t, j) : (x, i)(y, j) \in E(F)\}. For each \(x \in \mathbb{Z}_{12k+8}\) and each \(i \in \mathbb{Z}_2\) denote the vertex \((x, i)\) of \(\langle \{1, 2, 3^r\} \rangle_{12k+8}\) by \(x_i\).

The required 2-factorisation of \(\langle \{1, 2, 3^r\} \rangle_{12k+8}\) is given by the following five 2-factors.

1. \((0_0, 1_0, 0_1, 1_1) \cup (2_0, 3_0, 2_1, 4_0, 5_0, 3_1) \cup (4_1, 5_1, 7_0, 6_1, 7_1, 6_0) \cup (0_0, 1_0, 0_1, 2_0, 3_0, 1_1) + t \cup (2_1, 3_1, 4_1, 5_0, 6_0, 4_0) + t \cup (5_1, 6_1, 7_1, 8_0, 9_0, 7_0) + t \cup (8_1, 9_1, 11_0, 10_1, 11_1, 10_0) + t : t = 8, 20, 32, \ldots, 12k - 4\)

2. \((0_0, 2_0, 1_1, 2_1) \cup (1_1, 3_0, 4_0, 6_1, 5_1, 3_1) \cup (4_1, 5_0, 6_0, 7_0, 9_0, 7_1) \cup (0_0, 2_0, 1_1, 0_1, 3_0, 2_1) + t \cup (3_1, 4_0, 5_1, 4_1, 7_1, 5_0) + t \cup (6_0, 7_0, 6_1, 8_0, 10_1, 9_0) + t \cup (8_0, 9_1, 11_0, 13_0, 10_0, 11_0) + t : t = 8, 20, 32, \ldots, 12k - 4\)

3. \((0_0, 3_0, 1_0, 3_1) \cup (1_1, 2_0, 4_1, 6_1, 5_0, 2_1) \cup (4_0, 6_0, 5_1, 7_1, 8_1, 7_0) \cup (0_0, 3_0, 1_0, 2_1, 1_1, 3_1) + t \cup (2_0, 5_0, 7_0, 4_1, 6_0, 5_1) + t \cup (4_0, 6_1, 8_0, 10_1, 9_1, 7_1) + t \cup (8_1, 11_0, 9_0, 10_0, 12_1, 11_1) + t : t = 8, 20, 32, \ldots, 12k - 4\)

4. \((6_0, 9_0, 6_1, 9_1) \cup (0_1, 3_0, 5_1, 2_1, 4_1, 3_1) \cup (2_0, 4_0, 7_1, 8_0, 7_0, 5_0) \cup (0_1, 2_1, 5_0, 4_0, 2_0, 3_1) + t \cup (3_0, 4_1, 6_0, 9_0, 7_1, 5_1) + t \cup (6_0, 8_1, 7_0, 8_0, 10_0, 9_1) + t \cup (10_1, 12_0, 11_1, 13_1, 11_0, 13_0) + t : t = 8, 20, 32, \ldots, 12k - 4\)

5. \((6_0, 8_0, 6_1, 8_1) \cup (1_0, 2_0, 5_1, 4_0, 3_1, 2_1) \cup (3_0, 4_1, 7_0, 9_1, 7_1, 5_0) \cup (1_0, 2_0, 4_1, 2_1, 5_1, 3_1) + t \cup (3_0, 4_0, 7_0, 9_1, 6_1, 5_0) + t \cup (6_0, 7_1, 8_0, 11_1, 8_0) + t \cup (10_0, 12_0, 11_0, 12_1, 10_1, 13_1) + t : t = 8, 20, 32, \ldots, 12k - 4\)

We are now ready to prove our main result.

**Theorem 12** If \(F\) is a bipartite 2-regular graph of order \(rn\), then there exists a 2-factorisation of \(K_{r^n}\), \(r \geq 2\), into \(F\) if and only if \(n\) is even; except that there is no 2-factorisation of \(K_{6,6}\) into \([6,6]\).
Proof  A bipartite 2-regular graph has even order, so \( rn \) is even. Since a graph having a 2-factorisation is regular of even degree, if the 2-factorisation exists, then \( (r - 1)n \) (the degree of \( K_{nr} \)) is even. This together with the fact that \( rn \) is even implies that \( n \) is even when the 2-factorisation of \( K_{nr} \) exists, and it is known that there is no 2-factorisation of \( K_{6,6} \) into \([6,6] \), see [11] or [12].

Now, conversely, let \( n \) be even and let \( m = n/2 \) so that \( K_{nr} \cong K_{mr}^{(2)} \). If \( m \) is even or \( r \) is odd, then \( K_{mr} \) has even degree, and hence has a Hamilton decomposition by Theorem 1. So the result follows by Lemma 7 when \( m \) is even or \( r \) is odd. The result has been proved when \( r = 2 \) (see Theorem 3) and when \( n = 2 \) (see Theorem 2). Thus, we can assume \( m \geq 3 \) is odd and \( r \geq 4 \) is even.

By Lemma 8, there is a factorisation of \( K_{mr} \) into \( \frac{(r-1)m-3}{2} \) Hamilton cycles and a copy of \( \{1,3^r\}_{rm} \), and hence a factorisation of \( K_{mr} \cong K_{mr}^{(2)} \) into \( \frac{(r-1)m-3}{2} \) copies of \( C_{mr}^{(2)} \) and a copy of \( \{1,3^r\}_{rm}^{(2)} \). Each copy of \( C_{mr}^{(2)} \) can be factored into two copies of \( F \) by Lemma 7, and the copy of \( \{1,3^r\}_{rm}^{(2)} \) can be factored into three copies of \( F \) by Lemma 10; except when \( F \in \{[6^r], [4,6^r] : r \equiv 2 \mod 4 \} \). The case \( F = [6^r] \) with \( r \equiv 2 \mod 4 \) is covered by Theorem 4. Thus, the proof is complete except when \( r \geq 4 \) is even, \( m = \frac{n}{2} \geq 3 \) is odd, and \( F = [4,6^{4k+2}] \) for some \( k \geq 1 \) (where \( rm = 12k + 8 \)). We now deal with this special case.

By Lemma 9, there is a factorisation of \( K_{mr} \) into \( \frac{(r-1)m-5}{2} \) Hamilton cycles and a copy of \( \{1,2,3^r\}_{rm} \), and hence a factorisation of \( K_{mr} \cong K_{mr}^{(2)} \) into \( \frac{(r-1)m-5}{2} \) copies of \( C_{mr}^{(2)} \) and a copy of \( \{1,2,3^r\}_{rm}^{(2)} \). Each copy of \( C_{mr}^{(2)} \) can be factored into two copies of \( F \) by Lemma 7, and the copy of \( \{1,2,3^r\}_{rm}^{(2)} \) can be factored into five copies of \( F \) by Lemma 11. This completes the proof.  

We remark that the method used in the proof of Theorem 12 can also be used to obtain 2-factorisations in which the 2-factors are not all isomorphic. In the proof, distinct copies of \( C_{mr}^{(2)} \), and the copy of \( \{1,3^r\}_{rm}^{(2)} \) or \( \{1,2,3^r\}_{rm}^{(2)} \), can each be factored independently into specified 2-factors as described in Lemma 7, Lemma 10, and Lemma 11.
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