Observeing controlled state collapse in a single mechanical oscillator via a
direct probe of energy variance

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Due to their central role in our classical intuition of the physical world and their potential for interacting with
the gravitational field, mechanical degrees of freedom are of special interest in testing the nonclassical predictions
of quantum theory at ever larger scales. The projection postulate of quantum theory predicts that, for certain
types of measurements, continuously measuring a system induces a stochastic collapse of the state of the system
toward a random eigenstate. Here we propose an optomechanical scheme to observe this fundamental effect in
a vibrational mode of a mechanical membrane. The observation in the scheme is direct (it is not inferred via
an a priori assumption of the projection postulate for the mechanical mode) and is made possible through an
in situ probe of the mechanical energy variance. In the scheme, quantum theory predicts that a steady state
is reached as the measurement-induced collapse is counteracted by dissipation to the unmonitored environment.
Numerical simulations show this to result in a monotonic decrease in the time-averaged energy variance as the
ratio of continuous measurement strength to dissipation is increased. The measurement strength in the proposed
scheme is tunable in situ, and the behavior predicted by the simulations therefore implies a way to verifiably
control the time-averaged variance of a mechanical wave function over the course of a single quantum trajectory.
The scheme’s ability to directly probe the energy variance of the mechanical mode may also enable further
investigations of the effects on the mechanical state of coupling the mechanical mode to other quantum systems.

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I. BACKGROUND AND OVERVIEW

Quantum theory, whose predictions are manifest at micro-
scopic scales, contains no intrinsic prohibition for application
to macroscopic degrees of freedom. The manifestly classical
nature of the macroscopic world, however, makes such an
extrapolation far from trivial in significance. To be clear,
macroscopic phenomena such as superconductivity and crystal
structure were well understood to be direct manifestations of
quantum mechanics long ago. However, the possibility of a
macroscopic degree of freedom (i.e., one that is collective in
many microscopic degrees of freedom), such as the center
of mass of a crystal, itself exhibiting a classically forbidden
state or dynamical trajectory was mere speculation for several
decades after the advent of quantum theory (the Schrödinger
cat thought experiment is iconic of this). This changed with
the theoretical investigations of Leggett [1], which provided
momentum to a series of experiments in the 1980s with
collective electronic degrees of freedom in superconducting
circuits. These investigations culminated in the landmark
1988 experiment of Clarke et al. [2], which provided the
first unambiguous demonstration of the quantum tunneling
of a macroscopic degree of freedom (in this case, the phase
difference across a Josephson junction). Since then, microwave
cavity states [3], C$_{60}$ molecules [4], macroscopic currents [5,6],
and even a macroscopic mechanical dilution mode [7] have
all been demonstrated to occupy superpositions of classically
distinct states, clearly validating the Schrödinger equation for
macroscopic degrees of freedom.

Quantum-theoretical predictions, however, are sharply dis-
tinct from classical ones not only by way of the Schrödinger
equation, which dictates the behavior of a system in the
absence of measurement, but also through the projection
postulate, which applies in the scenario of measurement. In the
case of real finite-strength quantum measurements, in which
only partial information of an observable is extracted, the
projection postulate predicts that a measurement will result
in a partial, stochastic modification of the quantum state rather
than a complete collapse [8]. Observing this fundamental effect
requires a special class of measurements referred to as quantum
nondemolition (QND): a QND measurement of an observable,
which is possible for observables that commute with the
system Hamiltonian, leaves the postmeasurement state station-
ary (under the system Hamiltonian) in the eigenbasis of that
observable, and the difference between the pre- and postmea-
asurement states in this basis can therefore be attributed solely
to the effect of measurement. QND measurements have in fact
been used to successfully observe such measurement-induced
nonunitary quantum state evolution in the macroscopic degrees
of freedom of microwave cavities and superconducting qubits.
In [9], successive QND measurements on a microwave cavity
field initially prepared in a coherent state were used to infer the
progressive collapse of the coherent state toward a nearly pure
Fock state. Complete and permanent collapse to a pure state,
however, is never achievable in such a scenario due to unavoid-
able finite coupling to the unobserved environment; instead, if
measurement is continued after the collapse process, quantum
jumps between nearly pure Fock states arise, and these
were also observed in the same experiment. In [10–12] the
nonunitary modifications of a superconducting qubit state due
to QND measurements were observed, and the study in [13]
observed the progressive effect of continuous measurement on
the combined state of two qubits. Quantum jumps between the
ground and excited states of a superconducting qubit were first
observed in [14]. Regarding macroscopic mechanical degrees

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of freedom, however, no such experimental tests of nonunitary-state evolution due to measurement have been performed. Various theoretical investigations [15–26] have been done regarding proposals to observe, via continuous QND measurement, quantum jumps between nearly pure mechanical Fock states, but no proposals exist in the mechanical realm for experimental studies of the nonunitary-state collapse process itself.

In this theoretical work we propose a scheme to observe the measurement-induced progressive collapse of a mechanical wave function in the energy eigenbasis by directly monitoring its time-averaged variance in situ. The scheme is based on the platform of optomechanics (see [27–34] for reviews), wherein optical field modes are coupled to the motion of mechanical resonators. The particular system considered is the “membrane-in-the-middle” optomechanical system [35], which consists of a dielectric membrane suspended in the middle of an optical cavity and orthogonal to the cavity axis (see Fig. 1). Depending on the equilibrium position $x_0$ of the membrane along the cavity axis, the system can exhibit a modulation of the energy of a full cavity optical mode (having annihilation operator $a$) that could be either linear or quadratic in the mechanical membrane displacement $x$ along the cavity axis from $x_0$: $H \propto x^2 a + a^\dagger a$. In the quantum regime and under the rotating-wave approximation, the latter becomes $H \propto b^\dagger b a^\dagger a$ for a single mechanical mode (with annihilation operator $b$), thereby providing a channel for continuous QND measurement of mechanical energy: if mode $a$ is continuously driven with a fixed drive, the phase of its continuous output signal will depend on the mechanical energy, but mode $a$ does not exchange any quanta with mode $b$ and therefore does not perturb its energy. The linear-in-$x$ coupling provides a channel for actively cooling the mechanical mode to low occupation numbers [36,37]: driving mode $a$ at the red mechanical sideband induces a net up-conversion of the pumped photons via absorption of mechanical quanta. Further, tilting the membrane with respect to the cavity axis can change the optomechanical modulation of select optical spectra from $x^2$ to $x^4$ at lowest order in $x$ [38], providing a channel for QND measurements of $b^\dagger b + (b^\dagger b)^2$. Also, an examination of the full optical spectra of the system reveals that the $x$, $x^2$, and $x^4$ couplings may all be achieved simultaneously with independent optical channels. We show below that the capability of simultaneous $x$, $x^2$, and $x^4$ optomechanical coupling permits direct observation of the time-averaged variance (in the energy eigenbasis) of the quantum state of the mechanical mode while it is coupled to a thermal bath but actively cooled to the single-quantum regime.

The energy variance measurement scheme requires only an a priori assumption that the optical and mechanical modes, as well as their interaction, obey the Schrödinger equation, and that the Born rule applies to the optical modes. (The fully quantum nature of optical fields is well established by countless experiments, and the Schrödinger equation was first validated for a micron-scale mechanical degree of freedom in the experiment of [7].) The validity of the Schrödinger equation for optomechanical interactions was established in the recent experiment of [39], where the interaction was used to verifiably generate entanglement between a propagating microwave field and a micromechanical oscillator.) The projection postulate, thus far unvalidated for macroscopic mechanical degrees of freedom, predicts that the proposed scheme also permits in situ control of the time-averaged mechanical energy variance. As the measurement of the mechanical energy variance in the scheme does not entail an a priori assumption of the projection postulate, it serves as a legitimate test of this prediction. This predicted control of the energy variance is due to the interplay between a finite collapse rate of the quantum state (due to continuous QND measurement) and a finite broadening rate (due to continuous dissipation): the continuous QND measurement of $b^\dagger b$ through the $x^2$ optomechanical coupling produces the action of collapsing the quantum state toward a single (random) Fock state, while the coupling of the mechanical mode to dissipative channels induces a broadening of the quantum state toward a thermal state. The steady state between these two competing processes yields a finite time-averaged variance. Increasing the measurement strength on $b^\dagger b$, which may be done in situ by increasing the drive strength on the optical mode coupled to $x^2$, results in a smaller steady-state time-averaged variance because the collapse rate is thereby increased. Simultaneously, the information from the $x^4$ measurement channel may be combined with that from the $x^2$ channel to provide a direct observation of the steady-state time-averaged variance (see Sec. III). Thus, the collapse of a mechanical quantum state in the energy eigenbasis may be observed in a single time-averaged quantum trajectory by incrementally increasing the measurement strength after each sufficiently long time average of the measurement signals. The interplay of measurement-induced collapse and dissipation-induced broadening of the mechanical quantum state in this system was conceptually understood in a previous theoretical study that involved only measurements on $b^\dagger b$ [25]; here we present a proposal to experimentally observe and control this interplay over a range of relative strengths. The simultaneous mechanical mode cooling through the $x$ coupling serves the purpose of lowering the effective bath temperature of the mode, thereby reducing the $x^2$ coupling strength required to substantially collapse the quantum state.

The measurement-based collapse scheme outlined above should be contrasted with the fact that, for the Hamiltonian

\begin{align}
\omega_1 & = a_{1,L}^\dagger \omega_1 a_{1,L} + \omega_2 a_{2,L}^\dagger \omega_2 a_{2,L} + \omega_3 a_{3,L}^\dagger \omega_3 a_{3,L} \\
\omega_2 & = a_{1,R}^\dagger \omega_1 a_{1,R} + \omega_2 a_{2,R}^\dagger \omega_2 a_{2,R} + \omega_3 a_{3,R}^\dagger \omega_3 a_{3,R} \\
\omega_3 & = \frac{\omega_1 + \omega_2}{2} + \frac{\omega_1 - \omega_2}{2}
\end{align}
[Eq. (9)] of the system plus its environment, the Schrödinger equation by itself requires the mechanical quantum state to be a thermal state with a variance that slightly increases, rather than decreases, with increasing measurement strength (optical mode drive strength) [40].

It is also important to note that the previous state collapse investigations with microwave cavities and superconducting qubits mentioned above were all in the regime of sufficiently efficient measurements and negligible environmentally induced decay such that a significant fraction of the purity of the initial state was maintained over each quantum trajectory during the collapse process. By contrast, the scheme proposed here deals with macroscopic nonunitary quantum effects in during the collapse process. By contrast, the scheme proposed for efficient measurements and negligible environmentally in-

qubits mentioned above were all in the regime of sufficiently investigations with microwave cavities and superconducting

than decreases, with increasing measurement strength (optical

coupling for the fundamental mechanical mode $b$ so that the full Hamiltonian is given by

III. COLLAPSE OBSERVATION AND CONTROL

The protocol for observation and control of measurement-induced mechanical quantum state collapse is as follows. As mentioned in Sec. I, the projection postulate dictates that the steady-state mechanical quantum state under continuous measurement of $n_b$ is the result of a competition between collapse due to acquisition of information in the measurement record and broadening due to loss of information through the unmonitored dissipation channels. Although in this situation the quantum state itself fluctuates in time due to the continuous QND measurement, the long time average of its variance is constant. If the dissipation rates are constant, increasing the measurement strength on $n_b$ results in a smaller time-averaged variance for $n_b$. As the $n_b$ measurement strength is proportional to the drive on $a_{1+}$, the drive strength serves as an in situ experimental knob for the time-averaged variance. Selecting any values for the $a_{1+}$ drive strengths, one may take a long time average of both the output of $a_{1+}$ and its squared value to respectively extract $\langle n_{a_{1+}}(t) \rangle$ and $\langle n_{a_{1+}}^2(t) \rangle$, where the subscript $s$ denotes that $(\cdot)_{s} (t)$ is not an ensemble average but the mean value of the observable $u$ according to the single mechanical quantum state at time $t$. Simultaneously, one may obtain $\langle n_{a_{2+}}^2(t) \rangle + \langle n_{a_{1+}}^2(t) \rangle$ from a long time average of the output of $a_{2+}$ and combine it with the information from $a_{1+}$ to determine
\(\langle \sigma_\text{g}^2 \rangle_{1}(t)\). Thus, one obtains sufficient information to determine the time-averaged steady-state mechanical energy variance
\[\sigma_\text{g}^2(t) = \langle \sigma_\text{g}^2 \rangle_{1}(t) - \langle \sigma_\text{g} \rangle^2 \]
for each quantum trajectory at the selected values of the drive strengths. This experimental procedure requires no a priori assumption of the projection postulate. Repeating this procedure for incremented values of the drive on \(a_{1+}\), one may therefore test for the collapse of \(\sigma_\text{g}^2(t)\) with increasing measurement strength as predicted by the projection postulate. By relying on the measurements through the \(a_{1+}\) channel to collapse the quantum state, this protocol accommodates the fact that the experimentally observed \(g_2\) is very weak [38]; the information required from \(a_{2+}\) can always be obtained through sufficiently long time averages.

Being able to collapse the quantum state in this manner, however, implies certain parameter constraints. The study in [25] established two fundamental conditions for ensuring that the mechanical quantum state remained collapsed to a nearly pure Fock state so that quantum jumps would arise: both \(\kappa_{1+}\) (the damping rate for mode \(a_{1+}\)) and the \(n_b\) measurement rate \(\Gamma_1\) must be much greater than the mechanical Fock state decay rate. The collapse observation and control protocol in the present proposal therefore requires that both \(\kappa_{1+}\) and the maximum attainable value of \(\Gamma_1\) satisfy the same constraint.

The study in [25] considered the special case of a one-sided cavity where coupling to a thermal bath was the only source of mechanical dissipation. In the more realistic case that we consider here of a two-sided cavity with continuous sideband cooling that of the scheme entails that \(\pi_b\) be small. From detailed balance, \(\pi_b = (\gamma_{\text{cool}} \pi_{\text{opt}} + \gamma_b \pi_{\text{th}})/(\gamma_{\text{cool}} + \gamma_b)\). As the optical bath occupation \(\pi_{\text{opt}}\) is very small, choosing \(\gamma_{\text{cool}} \approx \gamma_{\text{b}}\pi_{\text{th}}\) yields \(\pi_b \approx 1\) and \(\Gamma_{1\text{max}} \approx 4\gamma_{\text{th}}\pi_{\text{th}}\). Thus, the steady state \(\pi_b\) can be achieved via continuous sideband cooling that is simultaneous with the collapse measurement and quantum jump measurement protocols without significantly increasing \(\Gamma_{1\text{max}}\) beyond what would be required in the absence of continuous sideband cooling, where the mechanical mode was instead passively cooled to \(\pi_b \approx 1\). Observing phonon-number quantum jumps and quantum state collapse with simultaneous sideband cooling may prove to be an experimentally more viable route than with passive cooling.

The authors of [22] derive the additional condition \(g_1^2 > \kappa_{1+}\) for detection of quantum jumps in energy, where \(\kappa_{1+}\) is the damping rate for mode \(a_{1+}\), by requiring that the phonon-number measurement rate be greater than the mechanical Fock state decay rate due to the Raman process mentioned above. However, because the measurement plays the dual role of detecting the Fock state and also collapsing the quantum state to create the Fock state, what is actually required is that the measurement rate be much greater than the Fock state decay rate. This was established in [25] and is reflected in the constraint on \(\Gamma_{1\text{max}}\) above. The true requirement is therefore
\[g_1^2 \gg \kappa_{1+}\]
Here, we assume that a mechanical mode energy state. For computational simplicity, we find as a test of the projection postulate in the mechanical realm. This is of importance for testing quantum theory at long time averages of the in situ mechanical Fock state. The photocurrents, now under the additional assumption that the Schrödinger equation applies to the mechanical mode and the optomechanical interaction, become

\[ i_1(t) = 2\eta\chi_1\langle n_b(t) \rangle + \sqrt{\eta\kappa_{1+}}\xi_1(t), \]

\[ i_2(t) = 2\eta\chi_2\langle n_b^2(t) + An_b(t) \rangle + \sqrt{\eta\kappa_{2+}}\xi_2(t). \]

Here, \( \Gamma_1 \) is the same phonon-number measurement rate discussed in the previous section. Experimentally, provided that \( A \) (defined above) is known, for fixed values of \( \Gamma_j \) sufficiently long time averages of \( i_1(t), i_2(t) \), and \( i_2(t) \) respectively yield the values of \( \langle n_b \rangle, \langle n_b^2 \rangle \) [45], and \( \langle n_b^2 \rangle \). As the derivation of the photocurrent expressions does not require an assumption of the projection postulate for the mechanical mode, the experimental acquisition of these values through the photocurrents can serve as a test of the projection postulate in the mechanical realm. The prediction of the projection postulate is that \( \sigma_\Gamma \) obtained from these experimental values will follow the monotonic behavior in Fig. 2, which is from simulations wherein the system density matrix is conditioned upon the measurement record. We remind the reader that \( \Gamma_j \) are proportional to \( a_j \), which can be adjusted in situ by varying the optical drive strengths.

Assuming \( T \approx 300 \) mK, \( \Omega \approx 2\pi \times 1 \) MHz [38,46], and \( \gamma_{th} = 2\pi \times 0.1 \) Hz [38], we find \( \bar{n}_b = 5 \times 10^3 \). As per above, we set \( \gamma_{cool} = \gamma_{th} \bar{n}_b \) so that \( \bar{n}_b \approx 1 \). Arbitrarily setting \( A = 1 \), we assume \( \kappa_{1+} \gg 4\gamma_{th} \bar{n}_b \) so that Eq. (14) is valid and we numerically integrate it for different values of \( \Gamma_1 \) to produce the data shown in Fig. 2.

Finally, we note that the environment is modeled here as a bath of noninteracting harmonic oscillators, but it may be that two-level systems also play an appreciable role in the mechanical dissipation [47,48]. This should not, however, affect the qualitative feature of a monotonic collapse, which simply depends on the generic effect of dissipation to an unmonitored environment.

V. CONCLUSION AND OUTLOOK

This work presents a feasible scheme to observe the measurement-induced collapse of a mechanical quantum state through a single time-averaged quantum trajectory. The proposed observation does not entail an \textit{a priori} assumption of the projection postulate for the mechanical quantum state and can therefore serve as a fundamental test of it in the mechanical realm. This is of importance for testing quantum theory at macroscopic scales. The state of the mechanical oscillator in the absence of measurement is a thermal state, a result of entanglement with its unmonitored environment [49,50], and the observable \textit{in situ} control of its variance via measurement may lead to further applications or other fundamental tests, as it is effectively a control of the amount of entanglement shared between the mechanical mode and its unmonitored environment. Also, setting the strengths of both measurement channels to be extremely weak can serve as a means of probing the time-averaged mechanical energy variance with very little measurement disturbance, and could serve as an \textit{in situ} means of probing the time-averaged effects on the mechanical energy variance of other quantum systems that may be coupled to the mechanical mode.
The scheme that we present is not too far from experimental reach as the latest iteration of the particular system considered shows orders of magnitude improvement in the $x^2$ optomechanical coupling strength [46]. A further order of magnitude increase in $g_1$ and optimization of $\kappa_{1\pm}$ should achieve the requirement $g_i^2 \gg \kappa_{1\pm}\kappa_{1\pm}$.

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**APPENDIX**

Below is a derivation of the model Hamiltonian $H_{\text{model}}$ in the main text. An explanation of the Raman scattering processes is contained in the final paragraph.

The full Hamiltonian is given by

$$H = \hbar \Omega b^\dagger b + H_1 + H_2 + H_{1, \text{drive}} + H_{2, \text{drive}} + H_{\text{diss}}, \quad \text{(A1)}$$

where

$$H_1 = H_1^{(0)} + H_1^{(\text{int})}, \quad \text{(A2)}$$

$$H_1^{(0)} = \hbar \omega_{j} a_{jL}^\dagger a_{jL} - \hbar \omega_{j} a_{jR}^\dagger a_{jR} + \hbar \omega_{j}, \quad \text{(A3)}$$

$$H_1^{(\text{int})} = -\hbar g_j (x/x_{\text{zpf}}) (a_{jL}^\dagger a_{jR} + a_{jR}^\dagger a_{jL}), \quad \text{(A4)}$$

and

$$H_2 = H_2^{(0)} + H_2^{(\text{int})}, \quad \text{(A5)}$$

$$H_2^{(0)} = \hbar \omega_2 a_{2L}^\dagger a_{2L} - \hbar \omega_2 a_{2R}^\dagger a_{2R} + \hbar \omega_2, \quad \text{(A6)}$$

$$H_2^{(\text{int})} = -\hbar g_2 (x/x_{\text{zpf}})^2 (a_{2L}^\dagger a_{2L} + a_{2R}^\dagger a_{2R}), \quad \text{(A7)}$$

and $H_{1, \text{drive}} + H_{2, \text{drive}} + H_{\text{diss}}$ is given in the main text. If $H_j^{(\text{int})}$ are treated as perturbations to $H_j^{(0)}$, the perturbations of the frequencies $\omega_{j\pm}$ can appear as a power series in $g_j$, and this is what we seek by using the approach of [26].

We first find the zeroth-order-in-$g$ time dependence of $x$ and $(a_{jL}^\dagger a_{jL} + a_{jR}^\dagger a_{jR})$ from the bare system Hamiltonian in the $a_{jL/R}$ basis:

$$H_{\text{sys}} = \hbar \Omega b^\dagger b + \sum_j (\hbar \omega_j a_{jL}^\dagger a_{jL} + \hbar \omega_j a_{jR}^\dagger a_{jR}) + \hbar g_j (x/x_{\text{zpf}})^2 (a_{jL}^\dagger a_{jL} - a_{jR}^\dagger a_{jR}) + \hbar J (a_{jL}^\dagger a_{jR} + H.c.) \quad \text{(A8)}$$

and then plug into $H_j^{(\text{int})}$ to find the force exerted on the mechanical mode:

$$F_j(t) = -\frac{\partial H_j^{(\text{int})}}{\partial x} \quad \text{(A9)}$$

For $j = 1$ this induces the forced mechanical motion

$$x_{1,1} = F_1(t)/m(\Omega^2 - 4 J_1^2), \quad \text{(A11)}$$

$$p_{1,1} = F_1(t)/(\Omega^2 - 4 J_1^2), \quad \text{(A12)}$$

while for $j = 2$ the forced mechanical motion is

$$x_{1,2} = \frac{1}{2 m} \left( \frac{F_2(t)}{\Omega^2 - (2 J_2 + \Omega^2)} \right)^2 + \frac{1}{2 m} \left( \frac{F_2(t)}{\Omega^2 - (2 J_2 - \Omega^2)} \right)^2, \quad \text{(A13)}$$

where $F_{2,\pm}(t) = 2 \hbar g j (a_{jL}^\dagger a_{jR} + a_{jR}^\dagger a_{jL}) e^{-i(\Omega t \pm \Omega \tau)} + H.c.$, $x_{i,j}$ and $p_{i,j}$ are the first-order-in-$g_j$ perturbations to the mechanical dynamics.

The next step is to make a time-dependent canonical transformation of the bare system Hamiltonian to a frame moving at the first-order-in-$g_j$ dynamics so that it cancels and the higher-order-in-$g_j$ perturbations become explicit:

$$e^{i(S_1+S_2)\hbar(\Omega b^\dagger b + H_1 + H_2)e^{-i(S_1+S_2)}}$$

$$= \hbar \Omega b^\dagger b + H_1^{(0)} + H_2^{(0)} + \frac{i}{2} \left[ S_1, H_1^{(\text{int})} \right] + \frac{i}{2} \left[ S_2, H_2^{(\text{int})} \right] + \frac{i}{2} \left[ S_2, H_2^{(\text{int})} \right] + O(g_j^2), \quad \text{(A15)}$$

where $S_j = x_{j,j} p / \hbar - p_{j,j} x / \hbar$, $x = x_{\text{zpf}}(b^\dagger + b)$, $p = i \hbar (b^\dagger - b) / x_{\text{zpf}}$, $x_{\text{zpf}} = \sqrt{\hbar / 2 m \Omega}$, and $m$ is the mechanical mode effective mass. Thus finding the expansion of the system Hamiltonian in powers of $g_j$, we plug it into the full Hamiltonian, transform to a picture moving with the zeroth-order optical and mechanical dynamics, make the rotating-wave approximation, and drop the first-order-in-$g_j$ terms of the expansion (as they do not modify the optical spectra) to find the effective Hamiltonian

$$H_{\text{eff}} = H_{1, \text{eff}} + H_{2, \text{eff}} + H_{1, \text{drive}} + H_{2, \text{drive}} + H_{\text{diss}} + O(g_j^2),$$

where

$$H_{1, \text{eff}}^{(2)} = \hbar \frac{g_1^2}{2} \left( \frac{4 J_1}{4 J_1^2 - \Omega^2} \right) (n_{1-} - n_{1+}) (1 + 2 n_b) + \hbar \frac{g_1^2}{2} \left( \frac{2 \Omega}{4 J_1^2 - \Omega^2} \right) (2 n_{1-} n_{1+} + n_{1+} + n_{1-}), \quad \text{(A16)}$$

$$H_{2, \text{eff}}^{(2)} = \hbar \frac{g_2^2}{2} \left( n_{2-} - n_{2+} \right) \left( A_2 n_b^2 + B n_b + C \right) - \hbar \frac{g_2^2}{2} \left( \frac{\Omega}{J_2 - \Omega} \right) n_{2-} n_{2+} (3 + 4 n_b), + \hbar \frac{g_2^2}{2} \left( \frac{\Omega^2}{J_2 - \Omega} \right) n_{2+} \quad \text{(A17)}$$
\( n_b = b^\dagger b, n_{\pm j} = a_{\pm j}^\dagger a_{\pm j} \), \( H'_{\text{drive}} = \hbar (\epsilon_j a_{\pm j} + \text{H.c.}) \), \( H_{\text{disc}} \) is unchanged as it is modeled with Lindbladian superoperators (see below) that are invariant under transformation to the zeroth-order dynamics, \( A_2 = \frac{\Omega}{2} + \frac{2\kappa_j^2}{\Omega^2 - \Omega_1}, B = \frac{\Omega}{2} + \frac{2\kappa_j^2 + \Omega_1}{\Omega^2 - \Omega_1} \), and \( C = \frac{\Gamma}{2} \frac{\Omega_1}{\Omega^2 - \Omega_1} + \frac{1}{2} + \frac{2\kappa_j^2}{\Omega^2 - \Omega_1} \). These expressions are valid for all values of the ratios \( \Omega / J_j \). We note that the coupling enhancement when \( 2J_1 - \Omega \ll J_1, \Omega \) noted in [26] for \( x^2 \) coupling has an analogous counterpart in the case of \( x^4 \) coupling when \( J_2 - \Omega \ll J_2, \Omega \), and if \( J_1 \gg \Omega \) the second lines of Eqs. (A16) and (A17) become negligible.

It is clear that the frequency of mode \( a_{1, \pm} \) is sensitive to \( n_b \) and the frequency of \( a_{2, \pm} \) is sensitive to both \( n_b \) and \( n_b^2 \). When the modes \( a_{1, \pm} \) are continuously driven, their outputs therefore yield continuous information on \( n_b \) and \( n_b^2 \). However, in the case that \( 2J_1 - \Omega \ll J_1, \Omega \), the frequency of \( a_{1, \pm} \) is actually sensitive to \( (n_{1, -} - n_b) \). Although we model the situation where the \( a_{1, \pm} \) are left undriven, \( H_{\text{int}}^{(\text{a})} \) support secondary Raman processes whereby quanta from \( a_{1, \pm} \) combine with phonons to scatter into \( a_{1, \pm} \). Letting \( \gamma_{1, \pm} \) be the total intrinsic dissipation rates of \( a_{1, \pm} \). Ref. [26] determines this to occur at a rate \( \chi_{1, \pm} \) to \( n_b \) to \( n_b^2 \) and \( n_{2, \pm} \), following model Hamiltonian:

\[
H_{\text{model}} = -\frac{\hbar}{2} g_1^2 A_1 n_{1, +} n_b - \frac{\hbar}{2} g_2^2 A_2 n_{2, \pm} (n_b^2 + \Lambda n_b) + H_{\text{drive}}^{(\text{a})} + H_{\text{disc}}^{(\text{a})} \]

where \( A_1 = (\frac{1}{\sqrt{2J_1 - \Omega}} + \frac{1}{\sqrt{\Omega_1 + \Omega}}), A = B / A_2 \), and dissipation due to the environment and Raman processes is contained in \( H_{\text{disc}} \).

The $b^\dagger b$ measurement strength is proportional to the number of optical quanta ($n_{1+}$) in the coupled optical mode. Increasing $n_{1+}$ decreases the effective mechanical oscillator frequency and therefore, if there is no measurement-induced collapse, should cause the mechanical energy variance to slightly increase, given a thermal bath at a constant temperature.

The decoherence theory of open quantum systems establishes that the density matrix of a system (here the mechanical mode) coupled to a macroscopic measurement apparatus (here the cavity mode) will be diagonal in the eigenbasis of the system observable (here the phonon number) that is coupled to the apparatus due to the environmentally induced rapid decay of the off-diagonal elements in that basis. In the present case, the mechanical mode density matrix will therefore be diagonal in the energy eigenbasis. For reviews of decoherence theory, see Refs. [43, 44].