Maximal representation dimension of finite $p$-groups

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Abstract. The representation dimension $\text{rdim}(G)$ of a finite group $G$ is the smallest positive integer $m$ for which there exists an embedding of $G$ in $\text{GL}_m(\mathbb{C})$. In this paper we find the largest value of $\text{rdim}(G)$, as $G$ ranges over all groups of order $p^n$, for a fixed prime $p$ and a fixed exponent $n \geq 1$.

1 Introduction

The representation dimension of a finite group $G$, denoted by $\text{rdim}(G)$, is the minimal dimension of a faithful complex linear representation of $G$. In this paper we determine the maximal representation dimension of a group of order $p^n$. We are motivated by a recent result of N. Karpenko and A. Merkurjev [7, Theorem 4.1], which states that if $G$ is a finite $p$-group then the essential dimension of $G$ is equal to $\text{rdim}(G)$. For a detailed discussion of the notion of essential dimension for finite groups (which will not be used in this paper), see [1] or [6, §8]. We also note that a related invariant, the minimal dimension of a faithful complex projective representation of $G$, has been extensively studied for finite simple groups $G$; for an overview, see [10, §3].

Let $G$ be a $p$-group of order $p^n$ and $r$ be the rank of the centre $Z(G)$. A representation of $G$ is faithful if and only if its restriction to $Z(G)$ is faithful. Using this fact it is easy to see that a faithful representation $\rho$ of $G$ of minimal dimension decomposes as a direct sum

$$\rho = \rho_1 \oplus \cdots \oplus \rho_r$$

(1)

of exactly $r$ irreducibles; cf. [8, Theorem 1.2]. Since the dimension of any irreducible representation of $G$ is $\leq \sqrt{|G:Z(G)|}$ (see, e.g., [11, Corollary 3.11]) and $|Z(G)| \geq p^r$, we conclude that

$$\text{rdim}(G) \leq rp^{\lfloor (n-r)/2 \rfloor}.$$  

(2)

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Let
\[ f_p(n) := \max_{r \in \mathbb{N}} \left( r \left( p^{(n-r)/2} \right) \right). \]

It is easy to check that \( f_p(n) \) is given by the following table:

<table>
<thead>
<tr>
<th>( n )</th>
<th>( p )</th>
<th>( f_p(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>even</td>
<td>arbitrary</td>
<td>( 2p^{(n-2)/2} )</td>
</tr>
<tr>
<td>odd</td>
<td>odd</td>
<td>( p^{(n-1)/2} )</td>
</tr>
<tr>
<td>odd, ( \geq 3 )</td>
<td>2</td>
<td>( 3p^{(n-3)/2} )</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

We are now ready to state the main result of this paper.

**Theorem 1.** Let \( p \) be a prime and \( n \) be a positive integer. For almost all pairs \((p, n)\), the maximal value of \( \text{rdim}(G) \), as \( G \) ranges over all groups of order \( p^n \), equals \( f_p(n) \). The exceptional cases are
\[
(p, n) = (2, 5), (2, 7) \text{ and } (p, 4), \text{ where } p \text{ is odd.}
\]

In these cases the maximal representation dimension is 5, 10, and \( p + 1 \), respectively.

The proof will show that the maximal value of \( \text{rdim}(G) \), as \( G \) ranges over all groups of order \( p^n \), is always attained for a group \( G \) of nilpotency class \( \leq 2 \). Moreover, if \((p, n)\) is non-exceptional, \( n \geq 3 \) and \((p, n) \neq (2, 3), (2, 4)\), the maximum is attained on a special class of \( p \)-groups of nilpotency class 2. We call these groups **generalized Heisenberg groups** since their representation theory looks very similar to that of the usual Heisenberg group (the group of unipotent upper triangular \( 3 \times 3 \) matrices); see §2.4

The rest of this paper is structured as follows. In §2 we introduce generalized Heisenberg groups and study their irreducible representations. In §3, we prove Theorem 1.

## 2 Generalized Heisenberg groups

### 2.1 Spaces of alternating forms.

Let \( V \) be a finite-dimensional vector space over an arbitrary field \( F \). Let \( \mathcal{A}(V) \) denote the space of bilinear alternating forms on \( V \); that is, linear maps \( b : V \otimes V \to F \) satisfying \( b(v, v) = 0 \).

Let \( K \) be a subspace of \( \mathcal{A}(V) \). Then \( K \) defines a map \( \omega_K : V \times V \to K^* \) as follows. Let \( j : \mathcal{A}(V)^* \to K^* \) denote the dual of the natural injection \( K \hookrightarrow \mathcal{A}(V) \). Then \( \omega_K \) is
defined to be the composition

\[ V \times V \to \Lambda^2(V) \to \mathcal{A}(V)^* \xrightarrow{j} K^*, \]  

where the first map is the natural projection and the second one is the canonical identification of \( \Lambda^2(V) \) with \( \mathcal{A}(V)^* \).

### 2.2 Symplectic subspaces.

**Definition 2.** A subspace \( K \subseteq \mathcal{A}(V) \) is **symplectic** if every non-zero element of \( K \) is non-degenerate, as a bilinear form on \( V \).

**Remark 3.** Equivalently, \( K \subseteq \mathcal{A}(V) \) is symplectic if and only if for every non-zero linear map \( K^* \to F \) the composition \( V \times V \xrightarrow{\omega_K} K^* \to F \) is non-degenerate.

Clearly non-trivial symplectic subspaces of \( \mathcal{A}(V) \) can exist only if \( \dim(V) \) is even.

**Lemma 4.** Suppose \( V \) is an \( F \)-vector space of dimension \( 2m \). If \( F \) admits a field extension of degree \( m \) then there exists an \( m \)-dimensional symplectic subspace \( K \subseteq \mathcal{A}(V) \).

**Proof.** Choosing a basis of \( V \), we can identify \( \mathcal{A}(V) \) with the space of alternating \( 2 \times 2 \)-matrices. Let \( f : M_m(F) \to \mathcal{A}(V) \) be the linear map

\[ A \mapsto \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix}. \]

If \( W \) is a linear subspace of \( M_m(F) = \text{End}_F(F^m) \) such that \( W \setminus \{0\} \subseteq \text{GL}_m(F) \) then \( K := f(W) \) is a symplectic subspace.

It thus remains to construct an \( m \)-dimensional linear subspace \( W \) of \( M_m(F) \) such that \( W \setminus \{0\} \subseteq \text{GL}_m(F) \). Let \( E \) be a degree \( m \) field extension of \( F \). Then \( E \) acts on itself by left multiplication. This gives an \( F \)-vector space embedding of \( \Psi : E \hookrightarrow \text{End}_F(E) \) such that \( \Psi(e) \) is invertible for all \( e \neq 0 \). \( \square \)

### 2.3 Groups associated to spaces of alternating forms.

Let \( V \) be a finite-dimensional vector space over a field \( F \). Let \( K \) be a subspace of \( \mathcal{A}(V) \) and let \( \omega_K \) denote the induced map \( V \times V \to K^* \), see (3). Choose a bilinear map \( \beta : V \times V \to K^* \) such that

\[ \omega_K(v, w) = \beta(v, w) - \beta(w, v). \tag{4} \]

To see that this can always be done, note that if \( \{e_i\} \) is a basis of \( V \), we can define \( \beta \) by

\[ \beta(e_i, e_j) = \begin{cases} \omega_K(e_i, e_j), & \text{if } i > j \\ 0, & \text{otherwise}. \end{cases} \]
We also remark that $\beta$ is uniquely determined by $\omega_K$, up to adding a symmetric bilinear form $V \times V \to K^*$.

**Definition 5.** Let $H = H(V, K, \beta)$ denote the group whose underlying set is $V \times K^*$ and whose multiplication is given by

$$(v, t) \cdot (v', t') = (v + v', t + t' + \beta(v, v')).$$

If $K$ is a symplectic subspace, we will refer to $H$ as a **generalized Heisenberg group**.

**Example 6.** Suppose $\omega$ is a non-degenerate alternating bilinear form on $V = F \oplus F$, where $F$ is a field of characteristic not equal to 2. Let $K$ be the span of $\omega$ in $\mathcal{A}(V)$. Then $H(V, K, \frac{1}{2}\omega)$ is isomorphic to the group of unipotent upper triangular $3 \times 3$ matrices over $F$. This group is known as the Heisenberg group.

**Remark 7.** It is easy to see that (5) is indeed a group law with the inverse given by $(v, t)^{-1} = (-v, -t + \beta(v, v))$ and the commutator given by

$$[(v_1, t_1), (v_2, t_2)] = (0, \omega_K(v_1, v_2)).$$

As $\omega_K$ is surjective, we see that $[H, H] = K^*$. Moreover, (6) also shows that $K^* \subset Z(H)$, and that equality holds unless the intersection \( \bigcap_{k \in K} \ker(k) \) is non-trivial. In particular, $Z(H) = K^*$ if $K$ contains a symplectic form.

**Remark 8.** A non-abelian finite $p$-group $S$ is called **special** if $Z(S) = [S, S]$ and $S/[S, S]$ is elementary abelian; see [4, §2.3]. Suppose $K$ is a subspace of $\mathcal{A}(V)$ such that $\bigcap_{k \in K} \ker(k)$ is trivial. Then over the finite field $\mathbb{F}_p$, the groups $H(V, K, \beta)$ are examples of non-abelian special $p$-groups. We are grateful to the referee for pointing this out.

**Remark 9.** If $\beta$ and $\beta'$ both satisfy (4) then $H(V, K, \beta)$ may not be isomorphic to $H(V, K, \beta')$. For example, let $V$ be a 2-dimensional vector space over $F = \mathbb{F}_2$, $K$ be the one-dimensional (symplectic) subspace generated by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $\beta$, $\beta'$ be bilinear forms on $V$ defined by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, respectively. Then $\beta$ and $\beta'$ both satisfy (4), but $H(V, K, \beta)$ is isomorphic to the quaternion group while $H(V, K, \beta')$ is isomorphic to the dihedral group of order 8.

On the other hand, it is easy to see that $H(V, K, \beta)$ and $H(V, K, \beta')$ are always isoclinic. (Recall that two groups $S$ and $T$ are isoclinic if there are isomorphisms $f : S/Z(S) \to T/Z(T)$ and $g : [S, S] \to [T, T]$ such that if $a, b \in S$ and $a', b' \in T$ with $f(aZ(S)) = a'Z(T)$ and $f(bZ(S)) = b'Z(T)$, then we have $g([a, b]) = [a', b']$, see [3].)

### 2.4 Representations.

Let $p$ be an arbitrary prime and let $F = \mathbb{F}_p$ be the finite field of $p$ elements. Fix, once and for all, a homomorphism $\tau : (\mathbb{F}_p, +) \hookrightarrow \mathbb{C}^*$. Let $W$ be
a finite-dimensional vector space over $F$. Using $\tau$, we identify the algebraic dual $W^* = \text{Hom}(W, F)$ with the Pontriyagin dual $\text{Hom}(W, C^*)$. It is clear that a bilinear alternating map $W \times W \to F_p$ is non-degenerate if and only if the composition $W \times W \to F_p \to C^*$ is non-degenerate.

Now let $V$ be a finite-dimensional vector space over $F$, $K$ a subspace of $\mathcal{A}(V)$, and $\omega = \omega_K$ the associated map. Choose $\beta$ satisfying (4) and let

$$G = H(V, K, \beta) = V \times K^*.$$

Recall that $K^*$ is in the centre of $G$ (Remark 7); in particular, it acts via a character on every irreducible representation of $G$.

**Lemma 10.** Let $\rho$ be a complex irreducible representation of $G$ such that $K^*$ acts by a character $\psi : K^* \to C^*$. Assume $\psi \circ \omega : V \times V \to C^*$ is non-degenerate.

(i) If $g \in G$, $g \notin K^*$, then $\text{Tr}(\rho(g)) = 0$.

(ii) $\dim(\rho) = \sqrt{|V|}$.

(iii) $\rho$ is uniquely determined (up to isomorphism) by $\psi$.

**Proof.** (i) Let $g \in G \setminus K^*$. Since $\psi \circ \omega$ is non-degenerate there exists $h \in G$ such that $\psi \circ \omega(gK^*, hK^*) \neq 1$. Observe that $\rho([g, h]) = \psi([g, h]) \text{Id}$, and that

$$\rho(h^{-1}gh) = \rho(g)\rho([g, h]).$$

Taking the trace of both sides, we have $\text{Tr}(\rho(g)) = \psi([g, h]) \text{Tr}(\rho(g))$. Since $\psi([g, h]) \neq 1$ we must have $\text{Tr}(\rho(g)) = 0$.

(ii) Since $\rho$ is irreducible, and the trace of $\rho$ vanishes outside of $K^*$, we have:

$$1 = \frac{1}{|G|} \sum_{g \in G} \text{Tr}(\rho(g)) \frac{\text{Tr}(\rho(g))}{|G|^2}$$

$$= \frac{1}{|G|} \sum_{g \in K^*} \text{Tr}(\rho(g)) \frac{\text{Tr}(\rho(g))}{|G|^2}$$

$$= \frac{1}{|G|} \dim(\rho)^2 \sum_{g \in K^*} \psi(g) \overline{\psi(g)}$$

$$= \dim(\rho)^2 \frac{|K^*|}{|G|}$$

Thus $\dim \rho = \sqrt{|G|/|K^*|} = \sqrt{|V|}$.

(iii) We have completely described the character of $\rho$, and it follows that $\rho$ is uniquely determined by $\psi$. Indeed,

$$\text{Tr}(\rho(g)) = \begin{cases} \sqrt{|V|} \cdot \psi(g), & \text{if } g \in K^* \\ 0, & \text{otherwise}. \end{cases}$$
In view of Remark 3, the following proposition is a direct consequence of the above lemma.

**Proposition 11.** The irreducible representations of a generalized Heisenberg group $H = H(V, K, \beta)$ are exhausted by the following list:

(i) $|V|$ one-dimensional representations, one for every character of $V$.
(ii) $|K| - 1$ representations of dimension $\sqrt{|V|}$, one for every non-trivial character $\psi : K^* \to \mathbb{C}^\times$.

The next corollary is also immediate upon observing that the centre of a generalized Heisenberg group $H = H(G, K, \beta)$ equals $K^*$; see Remark 7.

**Corollary 12.** The representation dimension of a generalized Heisenberg group $H = H(V, K, \beta)$ equals $\dim(K) \sqrt{|V|}$.

If $G$ is a finite Heisenberg group in the usual sense (as in Example 6) over $F = \mathbb{F}_p$, then for each non-trivial character $\chi$ of $Z(G)$ there is a unique irreducible representation $\psi$ of $G$ whose central character is $\chi$; cf. [2, §1.1]. This is a finite group variant of the celebrated Stone-von Neumann Theorem. For a detailed discussion of the history and the various forms of the Stone-von Neumann theorem we refer the reader to [9]. We conclude this section with another immediate corollary of Proposition 11 which tells us that over the field $\mathbb{F}_p$ every generalized Heisenberg group has the Stone-von Neumann property. This corollary will not be needed in the sequel.

**Corollary 13.** Two irreducible representations of a generalized Heisenberg group with the same non-trivial central character are isomorphic.

Corollary 13 is the reason we chose to use the term “generalized Heisenberg group” in reference to the groups $H(V, K, \beta)$, where $K$ is a symplectic subspace. Special $p$-groups (Remark 8) which are not generalized Heisenberg groups may not have the Stone-von Neumann property; see Remark 18.

### 3 Proof of Theorem 1

The case where $n \leq 2$ is trivial; clearly $\text{rdim}(G) = \text{rank}(G)$ if $G$ is abelian. We will thus assume that $n \geq 3$.

In the non-exceptional cases of the theorem, in view of the inequality (2), it suffices to construct a group $G$ of order $p^n$ with $\text{rdim}(G) = f_p(n)$. Here $f_p(n)$ is the function defined just before the statement of Theorem 1.

If $(p, n) = (2, 3)$ or $(2, 4)$, we take $G$ to be the elementary abelian group $(\mathbb{Z}/2\mathbb{Z})^3$ and $(\mathbb{Z}/2\mathbb{Z})^4$, yielding the desired representation dimension of 3 and 4, respectively. For all other non-exceptional pairs $(p, n)$, we take $G$ to be a generalized Heisenberg group over the field $\mathbb{F}_p$, as described in the table below. Here $H(V, K) \beta$ stands for $H(V, K, \beta)$, for some $\beta$ as in (4). In each instance, the existence of a symplectic subspace $K$ of suitable dimension is guaranteed by Lemma 4 and the value of $\text{rdim}(H(V, K))$ is given by Corollary 12.
This settles the generic case of Theorem 1. We now turn our attention to the exceptional cases. We will need the following upper bound on $\text{rdim}(G)$, strengthening (2).

Let $W_1(Z(G))$ be the subgroup of elements $g \in Z(G)$ such that $g^p = 1$.

**Lemma 14.** Let $G$ be a $p$-group and $r = \text{rank}(Z(G)) = \text{rank}(\Omega_1(Z(G)))$.

(i) Let $\rho_1$ be an irreducible representation of $G$ such that $\text{Ker}(\rho_1)$ does not contain $\Omega_1(Z(G))$. Then there are irreducible representations $\rho_2, \ldots, \rho_r$ of $G$ such that $\rho_1 \oplus \cdots \oplus \rho_r$ is faithful. In particular,

$$\text{rdim}(G) \leq \text{dim}(\rho_1) + (r - 1) \sqrt{|G : Z(G)|}.$$ 

(ii) If $\Omega_1(Z(G))$ is not contained in $[G, G]$, then

$$\text{rdim}(G) \leq 1 + (r - 1) \sqrt{|G : Z(G)|}.$$ 

The lemma can be deduced from [7, Remark 4.7] or [8, Theorem 1.2]; for the sake of completeness we give a self-contained proof.

**Proof.** (i) Let $\chi_1$ be the restriction to $\Omega_1(Z(G))$ of the central character of $\rho_1$. By our assumption $\chi_1$ is non-trivial. Complete $\chi_1$ to a basis $\chi_1, \chi_2, \ldots, \chi_r$ of the $r$-dimensional $\mathbb{F}_p$-vector space $\Omega_1(Z(G))^*$ and choose an irreducible representation $\rho_i$ such that $\Omega_1(Z(G))$ acts by $\chi_i$. (The representation $\rho_i$ can be taken to be any irreducible component of the induced representation $\text{Ind}_{\Omega_1(Z(G))}^G(\chi_i)$.) The restriction of

$$\rho := \rho_1 \oplus \cdots \oplus \rho_r$$

to $\Omega_1(Z(G))$ is faithful. Hence, $\rho$ is a faithful representation of $G$. As we mentioned in the introduction $\text{dim}(\rho_i) \leq \sqrt{|G : Z(G)|}$ for every $i \geq 2$, and part (i) follows.

(ii) By our assumption there exists a one-dimensional representation $\rho_1$ of $G$ whose restriction to $\Omega_1(Z(G))$ is non-trivial. Now apply part (i). \hfill $\square$

We are now ready to prove Theorem 1 in the three exceptional cases.

### 3.1 Exceptional case 1: $p$ is odd and $n = 4$.

**Lemma 15.** Let $p$ be an odd prime and $G$ be a group of order $p^4$.

(i) Then $\text{rdim}(G) \leq p + 1$.

(ii) Suppose $Z(G) \simeq (\mathbb{Z}/p\mathbb{Z})^2$ and $G/Z(G) \simeq (\mathbb{Z}/p\mathbb{Z})^2$. Then $\text{rdim}(G) = p + 1$. 

<table>
<thead>
<tr>
<th>$n$</th>
<th>$p$</th>
<th>$\text{dim}(V)$</th>
<th>$\text{dim}(K)$</th>
<th>$\text{rdim}(H(V, K))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>even, $\geq 6$</td>
<td>arbitrary</td>
<td>$n - 2$</td>
<td>2</td>
<td>$2p^{(n-2)/2}$</td>
</tr>
<tr>
<td>odd, $\geq 3$</td>
<td>odd</td>
<td>$n - 1$</td>
<td>1</td>
<td>$p^{(n-1)/2}$</td>
</tr>
<tr>
<td>odd, $\geq 9$</td>
<td>2</td>
<td>$n - 3$</td>
<td>3</td>
<td>$3p^{(n-3)/2}$</td>
</tr>
</tbody>
</table>
Proof. (i) We argue by contradiction. Assume there exists a group of order $p^4$ such that $\text{rdim}(G) \geq p + 2$. If $|Z(G)| > p^3$ or $G/Z(G)$ is cyclic then $G$ is abelian and $\text{rdim}(G) = \text{rank}(G) \leq 4 \leq p + 1$, a contradiction. If $Z(G)$ is cyclic then $\text{rdim}(G) \leq p$ by (2), again a contradiction.

Thus $Z(G) \cong G/Z(G) \cong (\mathbb{Z}/p\mathbb{Z})^2$. This reduces part (i) to part (ii).

(ii) Here $\Omega_1(Z(G)) = Z(G)$ has rank 2. Hence, a faithful representation $\rho$ of $G$ of minimal dimension is the sum of two irreducibles $\rho_1 \oplus \rho_2$, as in (1), each of dimension 1 or $p$.

Clearly $\text{dim}(\rho_1) = \text{dim}(\rho_2) = 1$ is not possible, since in this case $G$ would be abelian, contradicting $[G : Z(G)] = p^2$. It thus remains to show that $\text{rdim}(G) \leq p + 1$. Since $G/Z(G)$ is abelian, $[G, G] \subseteq Z(G)$. Hence, by Lemma 14(ii) we only need to establish that $[G, G] \subseteq Z(G)$.

To show that $[G, G] \subseteq Z(G)$, note that the commutator map

$$
\Psi : G/Z(G) \times G/Z(G) \to Z(G)
$$

$$
(gZ(G), g'Z(G)) \mapsto [g, g']
$$
can be thought of as an alternating bilinear map

$$(\mathbb{F}_p)^2 \times (\mathbb{F}_p)^2 \to (\mathbb{F}_p)^2.$$

Viewed in this way, $\Psi$ can be written as $\Psi(v, v') = (w_1(v, v'), w_2(v, v'))$ for $w_1, w_2 \in \mathbb{A}/((\mathbb{F}_p)^2)$. Since $\mathbb{A}/((\mathbb{F}_p)^2)$ is a one-dimensional vector space over $\mathbb{F}_p$, $w_1$ and $w_2$ are scalar multiples of each other. Hence, the image of $\Psi$ is a cyclic group of order $p$, and $[G, G] \subseteq Z(G)$, as claimed. □

To finish the proof of Theorem 1 in this case, note that $G = \mathbb{Z}/p\mathbb{Z} \times G_0$, where $G_0$ is a non-abelian group of order $p^3$, satisfies the conditions of Lemma 15(ii). Thus the maximal representation dimension of a group of order $p^4$ is $p + 1$, for any odd prime $p$.

3.2 Exceptional case 2: $p = 2$ and $n = 5$.

**Lemma 16.** Let $G$ be a group of order 32. Then $\text{rdim}(G) \leq 5$.

**Proof.** We argue by contradiction. Assume there exists a group of order 32 and representation dimension $\geq 6$. Let $r = \text{rank}(Z(G))$. Then $1 \leq r \leq 5$ and (2) shows that $\text{rdim}(G) \leq 5$ for every $r \neq 3$.

Thus we may assume $r = 3$. If $|Z(G)| \geq 16$ or $G/Z(G)$ is cyclic then $G$ is abelian, and $\text{rdim}(G) = \text{rank}(G) \leq 5$. We conclude that

$$Z(G) \cong (\mathbb{Z}/2\mathbb{Z})^3 \quad \text{and} \quad G/Z(G) \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

Applying the same argument as in the proof of Lemma 15(ii), we see that $[G, G] \subseteq Z(G)$, and hence $\text{rdim}(G) \leq 5$ by Lemma 14(ii), a contradiction. □
To finish the proof of Theorem 1 in this case, note that the elementary abelian group of order $2^5$ has representation dimension 5. Thus the maximal representation dimension of a group of order $2^5$ is 5.

### 3.3 Exceptional case 3: $p = 2$ and $n = 7$.

**Lemma 17.** If $|G| = 128$ then $\text{rdim}(G) \leq 10$.

**Proof.** Again, we argue by contradiction. Assume there exists a group $G$ of order 128 and representation dimension $\geq 11$. Let $r$ be the rank of $Z(G)$. By (2), $r = 3$; otherwise we would have $\text{rdim}(G) \leq 10$.

As we explained in the introduction, this implies that a faithful representation $\rho$ of $G$ of minimal dimension is the direct sum of three irreducibles $\rho_1$, $\rho_2$ and $\rho_3$, each of dimension $\sqrt{2^7/|Z(G)|}$. If $|Z(G)| > 8$, then

$$\dim(\rho_1) \leq 2 \quad \text{and} \quad \text{rdim}(G) = \dim(\rho_1) + \dim(\rho_2) + \dim(\rho_3) \leq 6,$$

a contradiction.

Therefore, $Z(G) \cong (\mathbb{Z}/2\mathbb{Z})^3$ and $\dim(\rho_1) = \dim(\rho_2) = \dim(\rho_3) = 4$. By Lemma 14(i) this implies that the kernel of every irreducible representation of $G$ of dimension 1 or 2 must contain $Z(G)$. In other words, any such representation factors through the group $G/Z(G)$ of order 16. Consequently, if $m_i$ is the number of irreducible representations of $G$ of dimension $i$ then $m_1 + 4m_2 = 16$. We can now appeal to [5, Tables I and II], to show that no group of order $2^7$ has these properties. From Table I we can determine which groups $G$ (up to isoclinism, cf. Remark 9) have $|Z(G)| = 8$ and using Table II we can determine $m_1$ and $m_2$ for these groups. There is no group $G$ with $|Z(G)| = 8$ and $m_1 + 4m_2 = 16$. □

We will now construct an example of a group $G$ of order $2^7$ with $\text{rdim}(G) = 10$. Let $V = (\mathbb{F}_2)^4$ and let $K$ be the 3-dimensional subspace of $\mathcal{S}(V)$ generated by the following three elements:

$$
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0
\end{bmatrix}.
$$

Let $G := H(V, K, \beta) = V \times K^\ast$ for some $\beta$ as in (4). Note that $K$ contains only one non-zero degenerate element (the sum of the three generators). In other words, there is only one non-trivial character $\chi$ of $K^\ast$ such that $\chi \circ \omega : V \times V \to \mathbb{C}^\times$ is degenerate. By Remark 7

$$[G, G] = Z(G) = K^\ast. \quad (7)$$

Let $\rho$ be a faithful representation of $G$ of minimal dimension. As we explained in the Introduction, $\rho$ is the sum of $\text{rank}(Z(G)) = 3$ irreducibles. Denote them by $\rho_1$, 

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$\varphi_2$, and $\varphi_3$, and their central characters by $\chi_1$, $\chi_2$ and $\chi_3$, respectively. Since $\rho$ is faithful, $\chi_1$, $\chi_2$ and $\chi_3$ form an $F_2$-basis of $\Omega_1(Z(G))^* \cong (Z/2Z)^3$. By Lemma 10, for each non-trivial character $\chi$ of $K^*$ except one, there is a unique irreducible representation $\psi$ of $G$ such that $\chi$ is the central character to $\psi$, and $\dim \psi = 4$. Thus at least 2 of the irreducible components of $\rho$, say, $\rho_1$ and $\rho_2$ must have dimension 4. By Lemma 17, $\dim(\rho) \leq 10$, i.e., $\dim(\rho_3) \leq 2$. But every one-dimensional representation of $G$ has trivial central character. We conclude that $\dim(\rho_3) = 2$ and consequently $\rho \dim(G) = \dim(\rho) = 4 + 4 + 2 = 10$.

Thus the maximal representation dimension of a group of order $2^7$ is 10.

**Remark 18.** The group $G$ constructed above has 16 one-dimensional representations with trivial central character, 4 two-dimensional representations with non-trivial degenerate central character, and 6 four-dimensional representations with pairwise distinct non-degenerate central characters. In view of (7), $H$ is a non-abelian special 2-group which does not enjoy the Stone-Von Neumann property (Corollary 13).

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**References**


