Detection loophole in Bell experiments: How postselection modifies the requirements to observe nonlocality

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A common problem in Bell-type experiments is the well-known detection loophole: if the detection efficiencies are not perfect and if one simply postselects the conclusive events, one might observe a violation of a Bell inequality, even though a local model could have explained the experimental results. In this paper, we analyze the set of all postselected correlations that can be explained by a local model, and show that it forms a polytope, larger than the Bell local polytope. We characterize the facets of this postselected local polytope in the Clauser-Horne-Shimony-Holt scenario, where two parties have binary inputs and outcomes. Our approach gives interesting insights on the detection loophole problem.

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I. INTRODUCTION

Quantum nonlocality, i.e., the fact that, in Bell’s terminology, no locally causal explanation can be given to quantum-mechanical correlations [1], is certainly one of the most fascinating and intriguing features of the quantum theory. Our classical understanding and apprehension of the physical world is quite disrupted by this characteristic, and experimental demonstrations are necessary for physicists and philosophers to accept such an upheaval.

A signature of nonlocality is the violation of a Bell inequality [1]. In the last 30 years, many Bell-type experiments have been performed to demonstrate quantum nonlocality [2], all of them showing good agreement with the quantum predictions. However, none of these experiments can be considered as perfectly convincing, as so far they all suffer from persistent loopholes: the skeptic can always find a (more or less far-fetched) classical explanation for the observed data. Given the crucial role of nonlocality in quantum information processing applications [3–6], loophole-free demonstrations of quantum nonlocality are highly desirable.

One of these loopholes is known as the detection loophole [7]. Typically, in photonic experiments the detection efficiencies are not perfect, and one usually postselects the detected events to show a violation of a Bell inequality. However, a model might exist that exploits the detector inefficiencies to reproduce the experimental data [7,8], in perfect agreement with Bell’s assumption of local causality [1]. In order to circumvent this problem, one usually resorts to the fair sampling assumption, that the detected particles are representative of all those emitted from the source, but this additional assumption is certainly not satisfactory. Closing the detection loophole would require either improving the detection efficiencies of the detectors used in Bell experiments, or finding Bell inequalities that are more robust to detection inefficiencies, as reported in [9–14]. Although the known necessary detection efficiencies are still quite high, a photonic detection-loophole-free Bell experiment seems possible in the near future.

Our goal here is to improve our understanding of the detection loophole problem and get a better intuition on it, by studying how postselection modifies the requirements for demonstrating nonlocality. We will show that the set of postselected local correlations is a polytope that includes the Bell local polytope (Sec. II). To illustrate this, we will consider the Clauser-Horne-Shimony-Holt (CHSH) scenario (Sec. III), with two parties both having two possible inputs and two outcomes (excluding the no-detection outcomes). This approach gives interesting insights on the (non-) locality of postselected correlations. It will allow us, in particular, to rederive and prove the optimality of Eberhard’s result on the tolerance of Bell tests to detection inefficiencies [15] in the CHSH scenario, and to understand why the quantum correlation that gives the largest violation of the CHSH inequality [16] is not the most robust to detection inefficiencies.

II. POSTSELECTED LOCAL CORRELATIONS

A. Bell-type experiment with imperfect detection efficiencies

Let us consider a typical Bell-type experiment involving two parties, Alice and Bob, with \( m_A \) and \( m_B \) inputs, and \( n_A \) and \( n_B \) outcomes, respectively.\(^1\)

If Alice and Bob have nonperfect detection efficiencies, we need to also take into account the possibility for Alice and Bob’s detectors not to fire (“∅”). They will thus actually have, respectively, \( n_A + 1 \) and \( n_B + 1 \) possible outcomes, denoted

\[
a = 1, \ldots, n_A, ∅, \quad b = 1, \ldots, n_B, ∅.
\]

After repeating the experiment many times, Alice and Bob can estimate their correlations, i.e., the probability distribution

\[
P_0(a,b|x,y)
\]

for \( a = 1, \ldots, n_A, ∅, \) \( b = 1, \ldots, n_B, ∅, \) and for the choice of measurement settings \( x = 1, \ldots, m_A \) and \( y = 1, \ldots, m_B \). We call \( P_0 \) the \textit{a priori} correlation: it is estimated before postselection. As it is standard in the study of nonlocality, we

\(^1\)In full generality, we could consider different numbers of outcomes for each observable, and a larger number of parties as well. The following study can easily be adapted to these cases.
will assume that \( P_0 \) is nonsignaling [i.e., \( P_0(a|x,y) = P_0(a|x) \) and \( P_0(b|x,y) = P_0(b|y) \)]; in an experiment, this can, in particular, be ensured by having Alice and Bob spacelike separated.

We will assume in the following that Alice’s and Bob’s detection probabilities are independent of their choice of measurement setting, and of what happens on the other partner’s side. Defining \( \eta_A \) (respectively, \( \eta_B \)) to be Alice’s (respectively, Bob’s) detection efficiency, this translates into the following constraints:

\[
\forall b,x,y, \quad P_0(a \neq \emptyset, b|x,y) = \eta_A P_0(b|y), \tag{3}
\]

\[
\forall a,x,y, \quad P_0(a, b \neq \emptyset | x,y) = \eta_B P_0(a|x), \tag{4}
\]

where we write \( P_0(a \neq \emptyset, b|x,y) = \sum_{a \neq \emptyset} P_0(a,b|x,y) \), \( P_0(a, b \neq \emptyset | x,y) = \sum_{b \neq \emptyset} P_0(a,b|x,y) \), and where we used the no-signaling assumption.

This implies, in particular, that (with obvious notations)

\[
\forall x,y, \quad P_0(a \neq \emptyset, b \neq \emptyset | x,y) = \eta_A \eta_B. \tag{5}
\]

### B. Postselected correlations

From their experimental data, Alice and Bob can postselect the conclusive events, when both detected their particle, and discard the nonconclusive events, as soon as one of the particles was not detected. They can thus estimate their postselected correlations, now for \( a = 1, \ldots, n_A \) and \( b = 1, \ldots, n_B \):

\[
P_{ps}(a,b|x,y) = \frac{P_0(a,b|x,y,a \neq \emptyset, b \neq \emptyset)}{P_0(a \neq \emptyset, b \neq \emptyset | x,y)}, \tag{6}
\]

i.e.,

\[
P_{ps}(a,b|x,y) = \frac{1}{\eta_A \eta_B} P_0(a,b|x,y). \tag{7}
\]

Note that the preceding independence assumption for \( \eta_A \) and \( \eta_B \) ensures that \( P_{ps} \) is also nonsignaling.

### C. Local causality assumption

In order for Alice and Bob to demonstrate nonlocality in their experiment, they need to check if their data before postselection can be explained by a local model.

The \( a \) \textit{a priori} correlation \( P_0(a,b|x,y) \) satisfies Bell’s standard local causality assumption \cite{L Bell 1964} if it can be decomposed in the form

\[
P_0(a,b|x,y) = \int \! d\lambda \; \rho(\lambda) \; P_0(a|x,\lambda) P_0(b|y,\lambda), \tag{8}
\]

for some local variables \( \lambda \) distributed according to \( \rho(\lambda) \). It is well known that the set of local correlations forms a convex polytope \cite{L Chsh 1969} (which we call, in our case here, the “local \( a \) \textit{a priori} polytope,” and denote by \( \mathcal{L}_0 \)), included in the polytope that contains all nonsignaling correlations (the “nonsignaling \( a \) \textit{a priori} polytope,” denoted by \( \mathcal{P}_0 \)).

Coming back to the postselected correlation \( P_{ps} \), we will say that it is “postselected local” if it can be obtained by postselecting the conclusive events of a local \( a \) \textit{a priori} correlation \( P_0 \), satisfying Eq. (8).

From Eqs. (3), (4), and (7), one can see that the set of postselected local correlations is, up to a factor \( \frac{1}{\eta_A \eta_B} \), the intersection of the local \( a \textit{priori} \) polytope \( \mathcal{L}_0 \), with the subspace defined by Eqs. (3) and (4). The intersection of a polytope with a subspace being a polytope \cite{L Branciard 2010}, the set of postselected local correlations is thus also a polytope, which we denote by \( \mathcal{L}_{ps}(\eta_A, \eta_B) \) (or simply \( \mathcal{L}_{ps} \) for short).

The postselected local polytope \( \mathcal{L}_{ps} \) clearly includes the local polytope \( \mathcal{L} \), which contains the local probability distributions for \( n_A \) and \( n_B \) inputs and \( n_A \) and \( n_B \) outcomes;\textsuperscript{2} both are included in the corresponding nonsignaling polytope \( \mathcal{P} \).\textsuperscript{3} However, there can be correlations in \( \mathcal{L}_{ps} \) that are not in \( \mathcal{L} \): these correlations will violate the standard Bell inequalities (which delimit the polytope \( \mathcal{L} \)) and therefore might “look nonlocal,” but they can still be explained by a local model with postselection.\textsuperscript{4}

Studying and characterizing the polytope \( \mathcal{L}_{ps}(\eta_A, \eta_B) \) allows one to understand which postselected correlations can or cannot be explained by a local model. This can easily be done once the polytope \( \mathcal{L}_0 \) has been characterized: indeed, the facets of \( \mathcal{L}_0 \) define, of course, valid inequalities for the intersection of \( \mathcal{L}_0 \) with the subspace defined by Eqs. (3) and (4); using Eqs. (3), (4), and (7), this leads to valid inequalities for the postselected local probabilities \( P_{ps} \in \mathcal{L}_{ps}(\eta_A, \eta_B) \). These inequalities are not all facets of \( \mathcal{L}_{ps} \), but since the polytope \( \mathcal{L}_{ps} \) is precisely delimited by the facets of \( \mathcal{L}_0 \), all of its own facets must be in the list of valid inequalities just obtained. Sorting all these inequalities thus allows one to extract all the facets of \( \mathcal{L}_{ps} \). In the following, we illustrate this in the CHSH scenario, where Alice and Bob have two possible inputs with binary outcomes (plus the no-detection events).

### III. POSTSELECTED LOCAL POLYTOPE \( \mathcal{L}_{ps}(\eta_A, \eta_B) \) IN THE CHSH SCENARIO

#### A. The standard CHSH scenario: Two inputs, two outcomes for Alice and Bob

The CHSH scenario corresponds to the simplest case, where Alice and Bob can both choose between two measurement settings, and have binary outcomes. In this case, all the nontrivial Bell inequalities that delimit the local polytope \( \mathcal{L} \)

\textsuperscript{2}Any local correlation \( P \in \mathcal{L} \) can indeed be turned into a local \( a \textit{priori} \) correlation \( P_0 \in \mathcal{L}_0 \) by just adding the possibility for Alice and Bob’s detectors not to fire, with independent probabilities \( \eta_A \) and \( \eta_B \). After postselection from \( P_0 \), we obtain back \( P_{ps} = P \), which proves that \( P \in \mathcal{L}_{ps} \). A similar argument allows one to show, more generally, that \( \mathcal{L}_{ps}(\eta_A, \eta_B) \subset \mathcal{L}_{ps}(\eta_{A1}, \eta_{B1}) \) for any \( \eta_A \leq \eta_{A1} \) and \( \eta_B \leq \eta_{B1} \). Note that \( \mathcal{L} = \mathcal{L}_{ps}(\eta_A = 1, \eta_B = 1) \).

\textsuperscript{3}The local and nonsignaling polytopes \( \mathcal{L} \) and \( \mathcal{P} \) should not be confused with the previous local and nonsignaling “\( a \textit{priori} \) polytopes” \( \mathcal{L}_0 \) and \( \mathcal{P}_0 \); the latter were indeed defined for correlations with \( n_A \) and \( n_B \) inputs, and \( n_A + 1 \) and \( n_B + 1 \) outcomes. In general, \( \mathcal{L}_0 \) and \( \mathcal{P}_0 \) are of dimension \((m_A + 1)(m_B + 1) - 1\), while \( \mathcal{L}_{ps} \) and \( \mathcal{P}_{ps} \) are of dimension \( [m_A(n_A - 1) + 1][m_B(n_B - 1) + 1] - 1 \) [19].

\textsuperscript{4}In fact, to conclude that these correlations are indeed nonlocal, one would usually resort to the fair sampling assumption; we do not want to use this additional assumption here.
are equivalent to the CHSH inequality\(^5\) [16,20], which can be written in the Clauser-Horne (CH) form [21] as
\[
P(1|1) + P(1|2) + P(1|21) - P(1|22) - P_A(1|1) - P_B(1|1) \leq 0, \quad \text{(9)}
\]
where \(P_A(a|x)\) [respectively, \(P_B(b|y)\)] denotes the marginal probability distribution of Alice (respectively, Bob).

It is convenient to use the notation introduced in [19], and write the CHSH (or CH) inequality as
\[
I_{CH} = \begin{pmatrix}
-1 & 0 \\
1 & 1 \\
0 & -1
\end{pmatrix} \leq 0, \quad \text{(10)}
\]
where the coefficients in the table are those that appear in front of the probabilities of getting the first outcome:
\[
P_A(1|1) \quad P_B(1|1) \\
P_A(1|2) \quad P_B(1|2)
\]
\[
P_A(1|2) \quad P(1|1) \quad P(1|2)
\]

B. The CHSH scenario with inefficient detectors:
Two inputs, three outcomes for Alice and Bob

In the case of inefficient detectors, there are now three possible outcomes on Alice and Bob's sides: \(\emptyset, 1,\) and 2.

The polytope \(\mathcal{L}_0\), corresponding to two inputs and three outcomes for both Alice and Bob, has been fully characterized in [19,22]. It has 1116 facets, of which 36 are trivial (simply corresponding to non-negative probabilities), 648 are of the CHSH form\(^6\) (with two outcomes grouped together on each side, so that Alice and Bob both have only two effective outcomes), and 432 are equivalent to the Collins-Gisin-Linden-Massar-Popescu (CGLMP) inequality [23]:
\[
I_{CGLMP} = \begin{pmatrix}
-1 & -1 & 0 \\
-1 & 1 & 0 \\
0 & 1 & -1
\end{pmatrix} \leq 0 \quad \text{(12)}
\]
(\(12\) facets & 2 columns)

C. Bell inequalities for \(\mathcal{L}_{ps}(\eta_A, \eta_B)\)

From the 1116 facets of \(\mathcal{L}_0\), and using Eqs. (3), (4), and (7), we obtain a list of valid inequalities for \(\mathcal{L}_{ps}(\eta_A, \eta_B)\) in the CHSH scenario. After sorting them, we find that, in addition to the trivial inequalities, it is actually sufficient to consider only the 64 equivalent forms of the following ones, as all the other inequalities are either trivial, or can be derived from them (see Appendix A1):
\[
-1 \leq I_{CH}^{\eta_A, \eta_B} \leq 0, \quad \text{(13)}
\]
with
\[
I_{CH}^{\eta_A, \eta_B} = \begin{pmatrix}
-\eta_A \\
\eta_A \eta_B \\
\eta_A \\
\eta_B \\
\end{pmatrix} \quad \text{(14)}
\]

Interestingly, the above inequalities are simply obtained from the CH inequality by grouping, for each observable, the outcome "\(y\)" with one of the other outcomes\(^7\) (see [15,24,25] for previous derivations of the inequality \(I_{CH}^{\eta_A, \eta_B} \leq 0\)). Here, the CGLMP inequality does not provide any additional Bell inequalities for the CHSH scenario with imperfect detectors.

We prove in Appendix A 2 that all the facets of \(\mathcal{L}_{ps}(\eta_A, \eta_B)\) are precisely, either of the trivial form \(P_{ps}(a,b|x,y) \geq 0\), or of the form \(I_{CH}^{\eta_A, \eta_B} \leq 0\) (if \(\eta_A + \eta_B < 3\eta_A \eta_B\)) or \(I_{CH}^{\eta_A, \eta_B} \geq -1\) [under a stronger constraint \(h(\eta_A, \eta_B) < 0\), with \(h\) defined in Eq. (A7)].

With this characterization, we now have the full list of all facets of \(\mathcal{L}_{ps}\); one can then easily check if a given correlation is postselected local or not. For that, Bell inequalities of the form (13) should be tested rather than the standard CHSH inequality (10).

D. Application:
Necessary conditions on \(\eta_A, \eta_B\) to observe nonlocality

One can now easily derive necessary conditions on \(\eta_A, \eta_B\) to observe nonlocality. Indeed, as proven in Appendix A 2, in order for \(\mathcal{L}_{ps}\) to have nontrivial facets, one must have
\[
\eta_A + \eta_B < 3\eta_A \eta_B. \quad \text{(15)}
\]
If this constraint is not satisfied, then only trivial inequalities delimit \(\mathcal{L}_{ps}\) (which is then actually equal to the full non-signaling polytope \(\mathcal{P}\)), and no violation can be observed.

In the symmetric case \(\eta_A = \eta_B = \eta\), we get the necessary condition
\[
\eta > \frac{2}{3}, \quad \text{(16)}
\]
which corresponds to the threshold obtained by Eberhard [15].

The condition (15), for general values of \(\eta_A\) and \(\eta_B\), had also been derived previously in [24]. For the special case \(\eta_A = 1\), we get the constraint \(\eta_B > \frac{1}{2}\) (see also [25]).

All these previous derivations [15,24,25] were based on the inequality \(I_{CH}^{\eta_A, \eta_B} \leq 0\). Our approach here allows us to justify this choice: we prove that this is, together with \(I_{CH}^{\eta_A, \eta_B} \geq -1\), the only relevant inequality in a CHSH scenario with imperfect detection efficiencies, but that \(I_{CH}^{\eta_A, \eta_B} \geq -1\) is less robust to detection inefficiencies.

\(^5\)Two inequalities are equivalent if they can be transformed into one another by relabeling the inputs, the outcomes, and/or exchanging the parties. In our case here, there are eight different equivalent versions of CHSH. The local polytope \(\mathcal{L}\) also has 16 other (equivalent) facets, which simply correspond to the non-negativity of the probabilities \(P(a,b|x,y)\); these facets, and the corresponding inequalities, are said to be trivial.

\(^6\)In fact, there are two inequivalent sets of 324 equivalent CHSH-like inequalities each.

\(^7\)Note that the two inequalities in Eq. (13) are, in general, nonequivalent, except if \(\eta_A = 1\) or \(\eta_B = 1\).

\(^8\)As proven in Appendix A 2, the inequality \(I_{CH}^{\eta_A, \eta_B} \geq -1\) is a facet of \(\mathcal{L}_{ps}\) only if \(h(\eta_A, \eta_B) < 0\), which is more restrictive than Eq. (15).
In comparison to previous proofs, here we really derived a necessary condition for observing nonlocality in a CHSH scenario, not only for observing a violation of a given inequality. To our knowledge, only the conditions $\eta > \frac{3}{4}$ in the symmetric case and $\eta_B > \frac{1}{2}$ in the special asymmetric case (with $\eta_A = 1$) were known to be necessary to observe nonlocality [11].

Finally, let us mention that this necessary condition is valid for all nonsignaling theories, and is not limited to quantum mechanics. Whether it is also a sufficient condition does, however, depend on the correlations one can achieve. It turns out that this is indeed the case for quantum correlations, which can violate $I_{\text{CH}}^{\eta_A,\eta_B} \leq 0$ for all $\eta_A, \eta_B$ such that $\eta_A + \eta_B < 3\eta_A\eta_B$ [24].

E. Geometric views

In order to get a better intuition, we now illustrate what the postselected local polytope $L_{\text{ps}}(\eta_A, \eta_B)$ looks like in some particular two-dimensional slices of the correlation space.

1. A nicely symmetric 2D slice

Let us first consider the two-dimensional (2D) slice that contains two (equivalent) PR boxes [26] $P_{\text{PR}}$ and $P_{\text{PR}'}$, and the fully random correlation $P_r$, defined as follows, in the notation of Eq. (11):

$$
\begin{align*}
P_{\text{PR}} &= \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 \end{pmatrix}, \\
P_{\text{PR}'} &= \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \\ 1/2 \end{pmatrix}, \\
P_r &= \begin{pmatrix} 1/2 \\ 1/4 \\ 1/4 \\ 1/4 \end{pmatrix}.
\end{align*}
$$

Any correlation in this slice can then be written in the form

$$P_{xy} = xP_{\text{PR}} + yP_{\text{PR}'} + (1 - x - y)P_r,$$

with $x,y \in \mathbb{R}$.

The trivial facets, the inequalities $I_{\text{CH}}^{\eta_A,\eta_B} \leq 0$ and $I_{\text{CH}}^{\eta_A,\eta_B} \geq -1$ (together with all their equivalent versions) respectively, impose the following constraints on $x$ and $y$ for $P_{xy}$ to be in $L_{\text{ps}}(\eta_A, \eta_B)$:

$$|x| + |y| \leq 1,$$

$$|x||y| \leq \frac{\eta_A + \eta_B - \eta_A\eta_B}{2\eta_A\eta_B} := F(\eta_A,\eta_B),$$

$$|x||y| \leq \frac{2 - \eta_A - \eta_B + \eta_A\eta_B}{2\eta_A\eta_B} := G(\eta_A,\eta_B).$$

Note that $\frac{1}{2} \leq F(\eta_A,\eta_B) \leq G(\eta_A,\eta_B)$, and therefore the last inequality above is implied by the previous one.

The structure of this two-dimensional slice, with these delimiting inequalities, is illustrated in Fig. 1.

The set $Q$ of quantum correlations corresponds in this slice to the disk $x^2 + y^2 \leq \frac{1}{2}$.\footnote{The constraint $x^2 + y^2 \leq \frac{1}{2}$ can be obtained from the criteria derived in [27] (see also [28]). The bound is tight, which can be seen as follows: consider the standard CHSH settings $\tilde{a}_1 = \tilde{z}, \tilde{a}_2 = \tilde{x}$, and $\tilde{b}_1, \tilde{b}_2 = \frac{\tilde{z}}{\sqrt{2}}$ (represented as vectors on the Bloch sphere), measured on the maximally entangled state $|\Phi^+\rangle$; we obtain the correlation $P_Q$, corresponding to $x = 0, y = \frac{\sqrt{2}}{2}$. Now, rotate the two settings of Bob together in the $xz$ plane of the Bloch sphere, and the whole circle $x^2 + y^2 = \frac{1}{2}$ is recovered.}

FIG. 1. (Color online) Two-dimensional slice of the correlation space corresponding to the CHSH scenario, containing the correlations $P_{\text{PR}}, P_{\text{PR}'}$, and $P_r$. $L_{\text{ps}}(\eta_A, \eta_B)$ is the thick blue polytope; the inner blue square delimits the local polytope $L = L_{\text{ps}}(1.1)$; the outer green diamond delimits the no-signaling polytope $P$; and the black circle corresponds to the set $Q$ of quantum correlations.

2. Illustration of Eberhard’s result

Eberhard’s result [15] that the correlation that gives a maximal violation of the standard CHSH inequality is not the most robust to detection inefficiencies, may seem surprising. Our approach here allows us to get a geometric intuition and a better understanding of this result.

As the detection efficiencies $\eta_A$ and/or $\eta_B$ decrease, the polytope $L_{\text{ps}}(\eta_A, \eta_B)$ continuously gets bigger, until it becomes equal to the full nonsignaling polytope $P$ when $\eta_A + \eta_B = 3\eta_A\eta_B$. Just before reaching the size of $P$, i.e., for $3\eta_A\eta_B - \eta_A - \eta_B$ just slightly positive, the last correlations that are non-"postselected local" are therefore to be found close to the boundaries of $P$; and as already mentioned, whatever $\eta_A, \eta_B$ such that $\eta_A + \eta_B < 3\eta_A\eta_B$, there exists quantum correlations in $P \setminus L_{\text{ps}}$ [24]. Clearly, the quantum correlation $P_Q$ is not
characterize in the CHSH scenario. In addition to providing Bell inequalities for postselected correlations, our approach allowed us, in particular, to give a necessary condition on the detection efficiencies to be able to observe nonlocality, and gave us a geometric intuition of the reason why the most robust correlation to detection inefficiencies is not the one that maximizes the violation of the standard CHSH inequality.

Note that our approach directly gives a characterization of $L_{ps}$ in terms of its facets; the difficulty, for larger numbers of inputs in particular, is to characterize the facets of $L_0$. Another possibility would be to first characterize the vertices of $L_{ps}$, and directly calculate its facets, without the need to evoke those of $L_0$. It is unclear to us whether there is a way to do this more efficiently than with our approach.

Be that as it may, we believe that our approach should motivate the study of Bell polytopes for scenarios with more inputs, but where Alice and/or Bob have three possible outputs that would correspond to binary outcomes plus the no-detection possibility. Even if the local $a$ priori polytope is not fully characterized, its known facets may imply nontrivial Bell-type inequalities for the corresponding postselected local polytope. In our study of the CHSH scenario, we found that all the facets of $L_{ps}$ could be obtained from those delimiting $\mathcal{L}$, by simply grouping the no-detection events with another outcome. However, this does not hold, in general, as we show in Appendix B. It would be interesting to find other cases where $L_{ps}$ has genuinely new facets compared to $\mathcal{L}$, and even find cases where these new facets can tolerate lower detection efficiencies to be violated.

Let us finally come back to the assumptions (3) and (4), that the detection efficiencies are independent of the choice of measurement settings. These assumptions were useful to carry out the present theoretical study, but might not be strictly satisfied in practical experiments. For practical purposes, one can either adapt our study to the observed situation, or simply avoid the detection loophole problem by not postselecting the conclusive events, and consider the full $a$ priori correlations directly.

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APPENDIX A: DETERMINING THE FACETS OF $L_{ps}(\eta_A, \eta_B)$ IN THE CHSH SCENARIO

1. Sorting all the valid inequalities for $L_{ps}$ obtained from the facets of $L_0$

From the 1116 facets of $L_0$ (as defined in the CHSH scenario with imperfect detection efficiencies), and using Eqs. (3), (4), and (7), we obtain a list of 1116 valid inequalities for $L_{ps}$ (some of them appearing several times). Note that because of

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10The asymmetric case $\eta_A = 1, \eta_B < 1$ is indeed of particular interest for experiments using atom-photon entanglement [24,25].
the particular role played by the no-detection outcome "∅," equivalent facets of \( L_0 \) do not necessarily define equivalent inequalities for \( \mathcal{L}_{ps}(\eta_A,\eta_B) \).

Most of these inequalities cannot be violated by any non-signaling correlations, and are simply implied by the non-negativity of the probabilities \( P(a,b|x,y) \). In addition to these trivial inequalities, we obtain three new inequalities (together with all their equivalent versions):

\[
-1 \leq I_{CH}^{\eta_A,\eta_B} \leq 0,
\]

with

\[
I_{CH}^{\eta_A,\eta_B} = \begin{pmatrix}
-\eta_B & 0 \\
\eta_A \eta_B & \eta_A \eta_B - \eta_A \eta_B \\
0 & \eta_A \eta_B - \eta_A \eta_B
\end{pmatrix},
\]

(A1)

with

\[
\begin{array}{c|cc}
\eta_A \eta_B & 0 & \eta_A(1-\eta_B) + \eta_B(1-\eta_A). \\
0 & \eta_A \eta_B - \eta_A \eta_B
\end{array}
\]

(A2)

The inequalities in Eq. (A1) can be obtained from the CHSH inequalities by grouping, for each observable, the outcome "∅" with one of the other outcomes; the inequality (A2) is obtained from the CGLMP inequality (12).

Interestingly, one can easily see that the inequality (A2) is actually implied by the upper bound in Eq. (A1):

\[
-\eta_A \eta_B & 0 \\
\eta_A \eta_B & \eta_A \eta_B - \eta_A \eta_B \\
0 & \eta_A \eta_B - \eta_A \eta_B
\]

\[
I_{CH}^{\eta_A,\eta_B} + \eta_A(1-\eta_B) 0 \\
0 0
\]

\[
\leq 0 + \eta_A(1-\eta_B) + \eta_B(1-\eta_A),
\]

(A3)

so it is actually sufficient to only consider the inequalities (A1).

2. Facets of \( \mathcal{L}_{ps}(\eta_A,\eta_B) \)

The polytope \( \mathcal{L}_{ps} \) is of dimension 8. To determine whether the remaining relevant inequalities are facets of \( \mathcal{L}_{ps}(\eta_A,\eta_B) \), we can try, for each inequality, to extract eight affinely independent correlations in \( \mathcal{L}_{ps} \) that saturate it.

a. Trivial facets

The 12 deterministic correlations \( P \) such that \( P(00|11) = 0 \) all saturate the trivial bound \( P(00|11) \geq 0 \), and are clearly in \( \mathcal{L}_{ps} \). Furthermore, one can easily extract eight of them that are independent. The inequality \( P(00|11) \geq 0 \) is therefore a facet of \( \mathcal{L}_{ps} \).

Equivalently, all the trivial inequalities \( P(a,b|x,y) \geq 0 \) are facets of \( \mathcal{L}_{ps} \).

b. Facets of the form \( I_{CH}^{\eta_A,\eta_B} \leq 0 \)

Using the decomposition

\[
I_{CH}^{\eta_A,\eta_B} = -(\eta_A + \eta_B - 3\eta_A \eta_B)P(11|11) + \eta_A(1-\eta_B)P(12|11) + \eta_B(1-\eta_A)P(21|11) + \eta_A \eta_B[P(12|12) + P(21|21) + P(11|22)],
\]

we first note that if \( \eta_A + \eta_B \geq 3\eta_A \eta_B \), then the inequality \( I_{CH}^{\eta_A,\eta_B} \leq 0 \) becomes trivial, since the coefficients in front of the probabilities in the above decomposition are all non-negative.

Let us then assume that \( \eta_A + \eta_B < 3\eta_A \eta_B \). Consider for instance the following correlations, written in the notation of Eq. (11), which all saturate the bound \( I_{CH}^{\eta_A,\eta_B} = 0 \):

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \\
1/2 & 1/2 & 0 & 1/2 & 0 & 1/2 & 0 & 1/2 & 0 & 1/2 & 0 \\
1/2 & 1/2 & 1/2 & x & x & y & y & y & , & y & y, \\
1/2 & 1/2 & 1/2 & x & x & y & y & y & , & y & y, \\
1/2 & 1/2 & 1/2 & x & 0 & y & y & y & , & y & y, \\
1/2 & 1/2 & 1/2 & x & y & y & y & , & y & y & y
\end{array}
\]

(A4)

where \( x = \frac{\eta_A + \eta_B - 2\eta_A \eta_B}{2\eta_A \eta_B} \) and \( y = \frac{(1-\eta_A)(1-\eta_B)}{\eta_A \eta_B(\eta_A \eta_B - 1)} \) (note that \( 0 \leq x, y < 1/2 \)).

These eight correlations are all in \( \mathcal{L}_{ps} \) (they satisfy all the inequalities that delimit \( \mathcal{L}_{ps} \)), and are independent. This proves that when \( \eta_A + \eta_B < 3\eta_A \eta_B \), the inequalities of the form \( I_{CH}^{\eta_A,\eta_B} \leq 0 \) are facets of \( \mathcal{L}_{ps}(\eta_A,\eta_B) \).

c. Facets of the form \( I_{CH}^{\eta_A,\eta_B} \geq -1 \)

Using the following decompositions:

\[
I_{CH}^{\eta_A,\eta_B} + 1 = \eta_A \eta_B[P(22|11) + P(11|12) + P(12|22)] + (1-2\eta_A)P(12|21) + (1-\eta_A)\eta_B P(22|21)
\]

\[
+ (1-\eta_A)(1-\eta_B)P_A(11|1) + (1-\eta_B)P_B(21|1)
\]

\[
(A5)
\]

\[
= (1-\eta_A)(1-\eta_B)P(11|11) + (\eta_A + \eta_B - 1)P(22|11) + (1-\eta_A)P(22|21) + (1-\eta_B)P(21|22)
\]

\[
+ (\eta_A \eta_B - \eta_A + \eta_B)P(11|12) + P(12|22)/2
\]

\[
+ (\eta_A \eta_B + \eta_A - \eta_B)P(11|21) + P(21|22)/2
\]

\[
+ (2 - \eta_A - \eta_B - \eta_A \eta_B)P(21|12) + P(12|21)/2,
\]

(A6)

we first note that if \( \eta_A + \eta_B + \eta_A \eta_B \leq 2 \), then the inequality \( I_{CH}^{\eta_A,\eta_B} \geq -1 \) becomes trivial:

(1) if \( \eta_A \leq \frac{1}{2} \), then the coefficients in front of the probabilities in the decomposition (A5) are all non-negative;

(2) if \( \eta_B \leq \frac{1}{2} \), then there exists a similar decomposition as Eq. (A5) with non-negative coefficients;

(3) if both \( \eta_A, \eta_B \geq \frac{1}{2} \) and if \( \eta_A + \eta_B + \eta_A \eta_B \leq 2 \), then the coefficients in front of the probabilities in the decomposition (A6) are all non-negative.
Let us refine the analysis in the case \( \eta_A + \eta_B + \eta_A \eta_B > 2 \), and define

\[
\begin{align*}
f_1(\eta_A, \eta_B) &= (1 - \eta_A)(1 - \eta_B)(3\eta_A \eta_B - \eta_A - \eta_B), \\
f_2(\eta_A, \eta_B) &= f_1(\eta_A, \eta_B) + 2(1 - \eta_A)^2 \eta_A^2, \\
g(\eta_A, \eta_B) &= f_1(\eta_A, \eta_B) + f_2(\eta_A, \eta_B) + f_3(\eta_B, \eta_A) \\
&\quad + 2(1 - \eta_A)(1 - \eta_B)(\eta_A + \eta_B - 2\eta_A \eta_B), \\
h(\eta_A, \eta_B) &= f_1(\eta_A, \eta_B) + f_2(\eta_A, \eta_B) + f_3(\eta_B, \eta_A) \\
&\quad - 2(\eta_A + \eta_B - 2\eta_A \eta_B)(3\eta_A \eta_B - 1).
\end{align*}
\]

We then have

\[
g(\eta_A, \eta_B)(I^{\eta_A, \eta_B}_{\text{CH}} + 1) \\
= (1 - \eta_A)(1 - \eta_B)h(\eta_A, \eta_B) - f_1(\eta_A, \eta_B)I^{\eta_A, \eta_B}_{\text{CH}} \\
- f_2(\eta_A, \eta_B)I^{\eta_A, \eta_B}_{ \text{CH}} - f_3(\eta_A, \eta_B)I^{\eta_A, \eta_B}_{\text{CH}} \\
+ 2 \eta_A \eta_B(1 - \eta_A)(1 - \eta_B)(\eta_A - 1)P(1222) \\
+ \eta_B(1 - \eta_A)P(2122) + (\eta_A + \eta_B - 2\eta_A \eta_B) \\
\times [P(2221)1 + P(2212) + P(2221)].
\]

(A7)

where \( I^{\eta_A, \eta_B}_{\text{CH}_1} \) and \( I^{\eta_A, \eta_B}_{\text{CH}_2} \), and \( I^{\eta_A, \eta_B}_{\text{CH}_5} \) are three equivalent versions of \( I^{\eta_A, \eta_B}_{\text{CH}} \), defined as

\[
\begin{align*}
I^{\eta_A, \eta_B}_{\text{CH}_1} &= \eta_A \eta_B - \eta_A - \eta_B - \eta_A \eta_B(1 - \eta_A) \eta_B, \\
I^{\eta_A, \eta_B}_{\text{CH}_2} &= -\eta_A \eta_B - \eta_A \eta_B(1 - \eta_A), \\
I^{\eta_A, \eta_B}_{\text{CH}_5} &= -\eta_A \eta_B - \eta_A \eta_B, \\
I^{\eta_A, \eta_B}_{\text{CH}_7} &= -\eta_A \eta_B - \eta_A \eta_B.
\end{align*}
\]

We thus see that when \( h(\eta_A, \eta_B) < 0 \) and \( \eta_A, \eta_B < 1 \), the inequality \( I^{\eta_A, \eta_B}_{\text{CH}} \geq -1 \) is no longer simply implied by the trivial and the \( I^{\eta_A, \eta_B}_{\text{CH}} \leq 0 \) facet inequalities. Since, from the previous analysis of Appendix A1, the facets of \( \mathcal{L}_{\text{ps}}(\eta_A, \eta_B) \) can only be of the trivial form, of the form \( I^{\eta_A, \eta_B}_{\text{CH}} \leq 0 \), or of the form \( I^{\eta_A, \eta_B}_{\text{CH}} \geq -1 \), we conclude that the inequalities \( I^{\eta_A, \eta_B}_{\text{CH}} \geq -1 \) are therefore, in this case, also facets of \( \mathcal{L}_{\text{ps}}(\eta_A, \eta_B) \).

Figure 3 illustrates the different cases that we studied, depending on the values of \( \eta_A \) and \( \eta_B \).

APPENDIX B: \( \mathcal{L}_{\text{ps}}(1, \eta) \) FOR \( m_A = 3, m_B = 2, n_A = n_B = 2 \)

Here we illustrate the fact that in general, the facets of \( \mathcal{L}_{\text{ps}} \) cannot all be derived from the facets of \( \mathcal{L} \) by just grouping the no-detection events with another outcome.

To show that, it suffices to allow a third possible input for Alice (\( m_A = 3, m_B = 2 \)), all observable still having binary outcomes (\( n_A = n_B = 2 \)), and to consider the case when Alice has perfect detectors (\( \eta_A = 1 \)), while Bob’s detection efficiency is \( \eta < 1 \).

We first note that in this scenario, the local polytope \( \mathcal{L} \) only has trivial facets and facets of the CH form, where one of Alice’s inputs is ignored [19].

The a priori polytope \( \mathcal{L}_0 \) corresponding to 3 and 2 inputs, 2 and 3 outputs for Alice and Bob, respectively, can be characterized using standard polytope algorithms [30]. It is found to have 1260 facets, 36 of which are trivial, 216 are of the CH form (with one of Alice’s inputs ignored, one of Bob’s outputs groups with another one), and respectively, 288, 288, and 432 of them are of the following forms:

\[
I^{3223(1)} = \begin{pmatrix} -1 & -1 & 0 & 0 \\
-1 & 1 & 1 & 0 \\
0 & 0 & 1 & -1 \end{pmatrix} \leq 0,
\] (B1)
Using similar methods as in Appendix A, one can show that the facets of the corresponding postselected local polytope $\mathcal{L}_p(1, \eta)$ are either of the trivial form, or, if $\eta > \frac{1}{2}$, of either one of the two forms

$$I_{\eta}^n \leq 0 \quad \text{or} \quad I_{\eta}^{323^{(n)}} \leq \eta,$$

with

$$I_{\eta}^{323^{(n)}} = \begin{bmatrix} -\eta & 0 \\ \eta & \eta' \\ 0 & \eta - \eta' \\ 0 & 0 \end{bmatrix}$$

The inequalities $I_{\eta}^n \leq 0$ can clearly be obtained from the CH inequalities that delimit $\mathcal{L}$, by grouping the no-detection events with another outcome. However, the inequalities $I_{\eta}^{323^{(n)}} \leq \eta$ cannot be obtained from the facets of $\mathcal{L}$, and are genuinely new. This is in contrast with what we observed in the CHSH scenario, where the facets of $\mathcal{L}_{ps}$ could all be derived from the facets of $\mathcal{L}$.

Note finally that for all $\eta > \frac{1}{2}$, both types of inequalities can be violated by quantum correlations. [To check that, one can consider for instance the following correlations: for $\eta > \frac{1}{2}$, define $X = \sqrt{\eta^2 - (1-\eta)^2}$, $\theta = \arcsin X$ and $|\psi\rangle = \sin \theta \left| 00 \right> + \cos \frac{\theta}{111}$; measuring $A_1 = \sigma_z, A_2 = \sigma_x, B_1 = \sigma_x + X\sigma_\eta, \sqrt{1+X^2}$, and $B_2 = \sigma_x + X\sigma_\eta$ on $\left| \psi \right>$ then gives $I_{\eta}^n \leq CH = \sqrt{\eta^2 - (1-\eta)^2} - \frac{1}{2} > 0$, while measuring $A_1 = \sigma_x, A_2 = \sigma_x - X\sigma_\eta, B_1 = \sigma_x, B_2 = \sigma_x$ on $\left| \psi \right>$ gives $I_{\eta}^{323^{(n)}} = \eta + \sqrt{\frac{\eta^2 - (1-\eta)^2}{2}} - \frac{1}{2} > \eta.]$