Quantum nondemolition measurements via quadratic coupling

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We consider making quantum nondemolition measurements on a harmonic oscillator by coupling it to an oscillator readout system using a four-wave-mixing interaction. The quantum nondemolition variable of interest appears as a nonlinear function of the number operator. It has the form $\hat{D}_1(t)=\cos(\chi a^\dagger a t)$. A sequence of measurements of this operator may be made with completely predictable results. The possible use of this system to detect gravitational radiation is suggessted.

I. INTRODUCTION

Recent research into possible gravitational radiation detection schemes has led to a consideration of how quantum fluctuations in the detection and associated amplification process limit the accuracy with which a classical force may be monitored. This is due to the fact that gravitational radiation reaching terrestial detectors is so weak as to produce displacements smaller than the quantum-mechanical uncertainties in the ground state of a harmonic oscillator. The detector must, then, be treated quantum mechanically. One is thus faced with the problem of making a sequence of measurements on a single quantum system, the results of which must be completely predictable, in the absence of the gravitational wave. For most dynamic variables this is not possible since intrinsic uncertainties in the results of a quantum-mechanical measurement, lead to fluctuations in the results of successive measurements. However, for certain detector variables, the so-called quantum nondemolition (QND) variables, it is possible.[1,2]

The condition for an operator $\hat{A}(t)$ to correspond to a QND observable as derived by Caves *et al.*[1,2] is

$$[\hat{A}(t'),\hat{A}(t)]=0 \; . \tag{1.1}$$

If this condition is satisfied then if the system begins in an eigenstate of $\hat{A}$ it remains in this eigenstate. Observables which are constants of the motion are clearly QND observables.

To measure the QND observable it is necessary to couple the detector to some amplifier or readout system. In the usual analysis of a QND measurement we treat the first stage of the readout system as another quantum system coupled to the detector. This system is referred to as the meter.

It is interesting that the meter need introduce no further fluctuations into the detector QND observable, provided that the only detector operator appearing in the detector-meter interaction Hamiltonian is the QND operator of the detector. This is the back-action evasion criteria. If this condition holds then $[\hat{A}(t'),\hat{A}(t)]=0$ in the presence of the interaction with the meter.

It should be noted that damping due to irreversible interactions with a heat bath will also limit the accuracy with which a QND observable may be measured.[3,4] In this paper both detector and meter are realized as harmonic oscillators.

An analysis of a QND measurement process may be divided into two stages. The first stage involves solving for the time-dependent unitary evolution of the coupled detector-meter system. During this stage correlations between the state of detector and meter build up. At some point the free evolution is suspended and a readout of the meter is made, whereupon the meter state is reduced. The second stage of the analysis then involves a determination of the nonunitary effect of meter-state reduction upon the detector.

In this paper, the interaction Hamiltonian is considered to be quadratic in the detector's quadrature phase amplitude. Unruh[5] has pointed out that such quadratic coupling schemes may realize quantum-counting QND measurements, which were among the earliest suggested QND measurement schemes.[6]

Unruh[7] has suggested a quadratic coupling scheme based on an LC circuit. We discuss a quadratic interaction model based on a quantum-optical four-wave-mixing interaction, with all modes treated quantum mechanically.

II. FOUR-WAVE-MIXING INTERACTION

We consider the coupled detector-meter system to be represented by the following Hamiltonian:

$$H=\hbar\omega_a a^\dagger a+\hbar\omega_b b^\dagger b+\hbar\chi a^\dagger a b^\dagger b \; , \tag{2.1}$$

where a (b) is an annihilation operator for the detector (meter) mode, ω_a and ω_b are the oscillator frequencies, and χ is the coupling constant.

If we define two operators by

$$\hat{X}_1(t) = \left[\frac{\hbar}{2\omega_a}\right]^{1/2}[a(t)+a^\dagger(t)] \, ,$$
$$\hat{X}_2(t) = \frac{1}{i}\left[\frac{\hbar}{2\omega_a}\right]^{1/2}[a(t)-a^\dagger(t)] \, , \tag{2.2}$$

corresponding to the real and imaginary parts of the detectors complex amplitude then

$$a^\dagger a = \frac{\hbar}{2\omega_a}(\hat{X}_1^2+\hat{X}_2^2)-\tfrac{1}{2} \, . \tag{2.3}$$

We thus see that the interaction Hamiltonian in (2.1) is quadratic in the complex amplitude. The oscillators could represent the quantized fields of two optical modes coupled by a third-order susceptibility as in four-wave mixing. However, since gravitational radiation produces mechanical oscillations in a detector we will refer to the quanta represented above as phonons.

It is clear from Eq. (2.1) that $a^\dagger a$ and $b^\dagger b$ are constants of motion. Since this is sufficient to satisfy the criteria for a QND variable we identify $a^\dagger a$ as a QND variable of the detector.[8] Furthermore, any function of $a^\dagger a$ will also be a QND variable. It also follows that the phonon number distribution for the detector (and meter) cannot change as a result of unitary evolution.

The solutions to the Heisenberg equations of motion in the interaction picture are

$$a(t) = \exp(-i\chi b^\dagger b t)a(0) \, ,$$
$$b(t) = \exp(-i\chi a^\dagger a t)b(0) \, , \tag{2.4}$$

where we have used the fact that $a^\dagger a$ and $b^\dagger b$ are constants of motion.

We now define two Hermitian operators $\hat{Y}_1(t)$ and $\hat{Y}_2(t)$ corresponding to the real and imaginary parts of the complex amplitude of the meter mode, respectively,

$$\hat{Y}_1(t) = \left[\frac{\hbar}{2\omega_b}\right]^{1/2}[b(t)+b^\dagger(t)] \, ,$$
$$\hat{Y}_2(t) = \frac{1}{i}\left[\frac{\hbar}{2\omega_b}\right]^{1/2}[b(t)-b^\dagger(t)] \, . \tag{2.5}$$

Using Eqs. (2.4) and (2.5) we obtain

$$\hat{Y}_1(t) = \hat{D}_1(t)\hat{Y}_1(0)+\hat{D}_2(t)\hat{Y}_2(0) \, , \tag{2.6}$$

where we define two new operators by

$$\hat{D}_1(t) = \cos(\chi a^\dagger a t) \, ,$$
$$\hat{D}_2(t) = \sin(\chi a^\dagger a t) \, . \tag{2.7}$$

We note that $\hat{D}_1(t)$ and $\hat{D}_2(t)$ are also QND operators. Equation (2.6) allows us to infer values of $\hat{D}_1(t)$ from measurements of $\hat{Y}_1(t)$.

If we now prepare the initial state of the detector in a coherent state such that $\langle \hat{Y}_2(0)\rangle = 0$ and $\langle \hat{Y}_1(0)\rangle = (2\hbar/\omega_b)^{1/2} y_1(0)$ we can infer a value for $\hat{D}_1(t)$ by making measurements on $\hat{Y}_1(t)$. A measurement of $\hat{Y}_1(t)$ gives a result $y_1(t)$ from which the experimenter infers a value $d_1(t)$ for $\hat{D}_1(t)$ given by

$$d_1(t) = \left[\frac{\omega_b}{2\hbar}\right]^{1/2}\frac{y_1(t)}{y_1(0)} \, . \tag{2.8}$$

The error in this value $\Delta d_1(t)$ is then given by

$$\Delta d_1(t) = \left[\frac{\omega_b}{2\hbar}\right]^{1/2}\frac{\Delta\hat{Y}_1(t)}{y_1(0)} \, , \tag{2.9}$$

where $[\Delta\hat{Y}_1(t)]^2$ is the variance in $\hat{Y}_1(t)$ at the time of measurement.

If the meter is initially in a coherent state $|y_1(0)\rangle$ we find that

$$[\Delta Y_1(t)]^2 = \frac{\hbar}{2\omega_b}+\frac{2\hbar}{\omega_b}y_1^2(0)[\Delta\hat{D}_1(t)]^2 \tag{2.10}$$

then

$$\Delta d_1(t) = \left[\frac{1}{4y_1^2(0)}+[\Delta\hat{D}_1(t)]^2\right]^{1/2} \, . \tag{2.11}$$

We can see that it is impossible to predict with certainty the outcome of a measurement of $\hat{D}_1(t)$ if it is very uncertain at the time of readout. However, if $[\Delta\hat{D}_1(t)]^2$ is small then by preparing the meter in a highly excited coherent state [$y_1(0)$ is large] we can make our determination of $d_1(t)$ as certain as we like.

Before proceeding to the second step in a QND measurement analysis, meter-state reduction, we first discuss some properties of the QND variable $\hat{D}_1(t)$. [Similar statements can be made concerning $\hat{D}_2(t)$.]

The eigenstates of $\hat{D}_1(t_1)$ are clearly number states $|n\rangle$ with eigenvalue $d_1(t_1)$ given by

$$d_1(t_1) = \cos(\chi n t_1) \, . \tag{2.12}$$

However, due to the periodic nature of the cosine function this eigenvalue is degenerate. The eigenstates $|n\rangle$, where

$$n = \left|\frac{2\pi l}{\chi t_1} \pm \frac{\cos^{-1}[d_1(t_1)]}{\chi t_1}\right| \quad l = 0,1,2,\ldots \tag{2.13}$$

(and n is restricted to an integer) all have the same eigenvalues.

It should be noted, however, that while the number states of Eq. (2.13) are also eigenstates of $\hat{D}_1(t_2)$ ($t_2 \neq t_1$), these states are no longer degenerate. Acting on (2.13) with $\hat{D}_1(t_2)$ we have

$$\hat{D}_1(t_2)\left|\frac{2\pi l}{\chi t_1}\pm\frac{\theta}{\chi t_1}\right\rangle = d_1(t_2)\left|\frac{2\pi l}{\chi t_1}\pm\frac{\theta}{\chi t_1}\right\rangle , \tag{2.14}$$

where $\theta=\cos^{-1}[d_1(t_1)]$ and $d_1(t_2)=\cos[(\theta+2\pi l)t_2/t_1]$. These eigenvalues are dependent on the value of l. Thus, while a linear superposition of the states in Eq. (2.13) is also an eigenstate of $\hat{D}_1(t_1)$, it is not an eigenstate of $\hat{D}_1(t_2)$.

The periodicity of the cosine function also implies that if the system were in the pure number state $|n\rangle$, a single measurement of $\hat{D}_1(t_1)$ could not be used to infer a unique value for n. However, a second measurement of $\hat{D}_1$ at time t_2, such that $\chi(t_1-t_2)$ is not a rational number, allows n to be inferred uniquely from the results of the two measurements. This is a consequence of n being an integer. In any practical scheme, of course, it is only possible to choose $\chi(t_1-t_2)$ to a fixed number of decimal places and it may be impossible to ensure that $n\chi(t_1-t_2)$ is not an integer multiple of 2π over the range of n values appropriate to the detector description. We will not further pursue schemes to make a precise determination of n.

An instantaneous perfectly accurate measurement of $\hat{D}_1(t)$ with the result $d_1(t)$ will leave the system in a linear combination of eigenstates corresponding to this measured eigenvalue. Which linear combinations occur depend on which eigenstates were present in the premeasurement number distribution of the detector. Once the measurement is made and the detector state has "collapsed" to this linear combination of number states, the corresponding phonon number distribution cannot change as a result of free evolution of the coupled system. A subsequent (perfectly accurate) measurement of $\hat{D}_1(t)$ must then yield the same result $d_1(t)$, providing the measurement is made after the same elapse of time. This is what is required of a QND measurement.

If the variance of $\hat{D}_1(t)$ is zero at any time, it periodically returns to zero. This is a consequence of the fact that $\hat{D}_1(t)$ is a stroboscopic QND variable.[1] In Fig. 1 we have plotted the variance of $\hat{D}_1(t)$ as a function of time for an initial coherent state. The stroboscopic nature of $\hat{D}_1(t)$ is clearly evident. Thus if the system at one time is in an eigenstate of $\hat{D}_1$ it will periodically return to that eigenstate.

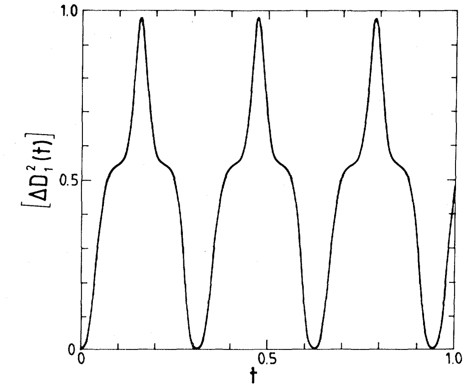

FIG. 1. Variance in $P_1(t)$ vs time for a detector in a coherent state with $\bar{n}=1.0$.

III. METER-STATE REDUCTION

We now proceed to the second step in a QND measurement analysis, a determination of the nonunitary effect of meter-state reduction when a readout is made. We use the projection-operator technique of Caves *et al.*[1]

Let the state of the detector before measurement be characterized by the phonon number distribution $P(n)$. The initial density operator for the total detector-meter system may be written as

$$\rho(0)=|i_M\rangle|i_D\rangle\langle i_D|\langle i_M| , \tag{3.1}$$

where $|i_D\rangle$ ($|i_M\rangle$) is the initial state of the detector (meter). Thus

$$P(n)=|\langle i_D|n\rangle|^2 . \tag{3.2}$$

In the Schrödinger picture the density operator at time t is given by

$$\rho(t)=\exp\left[-\frac{i}{\hbar}Ht\right]\rho(0)\exp\left[\frac{i}{\hbar}Ht\right] . \tag{3.3}$$

At time t_1 we read out a value for $\hat{Y}_1(t_1)$ with the result $y_1(t_1)$ and project the meter into an eigenstate of $\hat{Y}_1(t_1)$ with an eigenvalue equal to this measured result.

The detector state after readout is then characterized by the following density operator:

$$\rho_D(t_1)_R = N\langle y_1(t_1),t_1|e^{-iHt_1/\hbar}\rho(0)e^{iHt_1/\hbar}|y_1(t_1),t_1\rangle , \tag{3.4}$$

where $|y_1(t_1),t\rangle$ is an eigenstate of $\hat{Y}_1(t_1)$ with eigenvalue $y_1(t)$ (and N is some normalization constant). The phonon number distribution of the detector after readout is then given by

$$P(n)_R = N|\langle y_1(t_1),t_1|\exp[-i(\omega_b+\chi n)b^\dagger b t_1]|i_M\rangle|^2 P(n) . \tag{3.5}$$

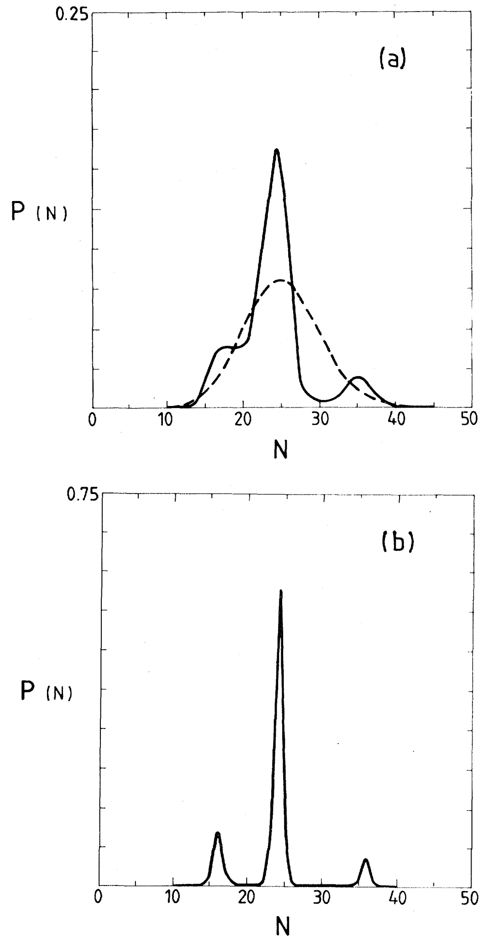

FIG. 2. Phonon number distribution after readout (solid line) for an initial coherent state distribution (dashed line). Two different meter states are considered: (a) $y_1^2(0)=2.0$ and (b) $y_1^2(0)=15.0$. In both (a) and (b), $f_1(t)=0.5$ and $t=0.1\pi$.

Let us now assume the detector is initially in a coherent state $|y_1(0)\rangle$ with $y_1(0)$ real. Then $\langle \hat{Y}_2(0)\rangle=0$ and

$$\langle \hat{Y}_1(0)\rangle=\left[\frac{2\hbar}{\omega_b}\right]^{1/2} y_1(0) . \tag{3.6}$$

The eigenstates of $\hat{Y}_1(t)$ are obtained from the eigenstates of $\hat{Y}_2(0)$ by[1]

$$|y_1(t),t\rangle=\exp(-i\omega_b b^\dagger bt)\,|y_1(t),0\rangle . \tag{3.7}$$

We next note that $\hat{Y}_1(0)=\hat{x}$, the coordinate operator, and thus $|y_1(t),0\rangle$ is a position eigenstate. We are now in a position to evaluate Eq. (3.5). To do this we expand $|y_1(t),0\rangle$ in terms of number states[9]

$$|y_1(t),0\rangle=\left[\frac{\omega_b}{\pi\hbar}\right]^{1/4} e^{-\omega_b[y_1(t)]^2/2\hbar} \times \sum_{l=0}^{\infty}\frac{H_l[(\omega_b/\hbar)^{1/2}y_1(t)]}{2^{l/2}\sqrt{l!}} , \tag{3.8}$$

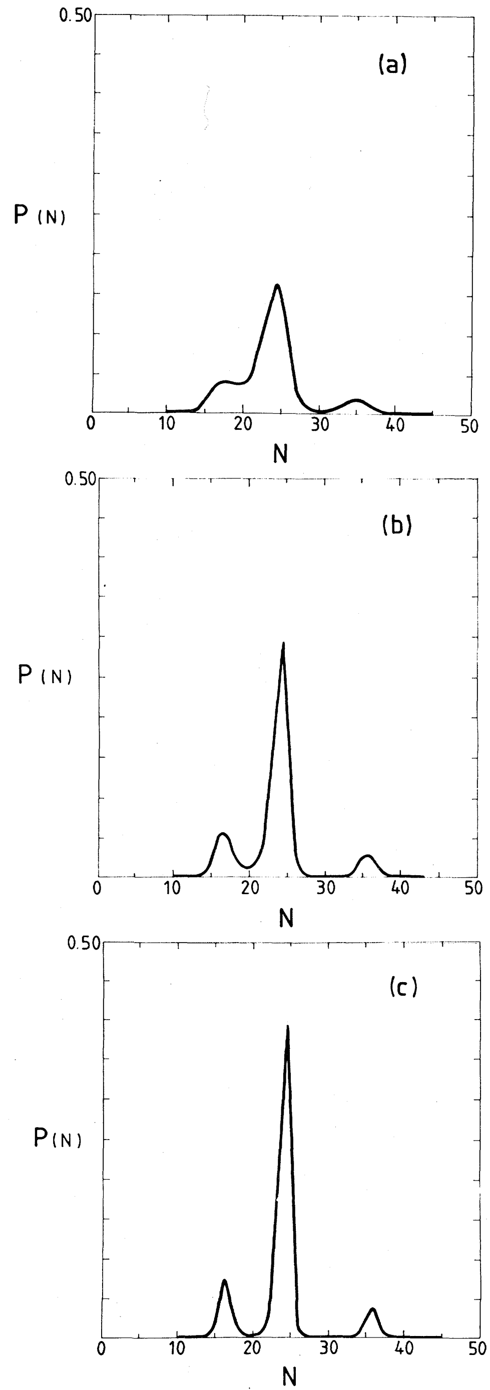

FIG. 3. Phonon number distribution after three successive readouts (a,b,c) for an initial coherent state. $y_1^2(0)=1.0$, $f_1(t_1)=0.3$, $\chi t_1=0.1\pi$.

where $H_l(x)$ is a Hermite polynomial of order l. We also expand the coherent state $|y_1(0)\rangle$ in terms of number states. If we then use the following identity[10]:

$$\sum_{l=0}^{\infty} \frac{a^l}{2^l l!} H_l(x) = \exp\left[ax - \frac{a^2}{4}\right] , \tag{3.9}$$

Eq. (3.5) may be written as

$$P(n)_R = N \exp\left\{-2\left[y_1(0)\cos(\chi n t_1) - \left[\frac{\omega_b}{2\hbar}\right]^{1/2} y_1(t_1)\right]^2\right\} P(n) . \tag{3.10}$$

Using Eqs. (2.8) and (3.6) Eq. (3.10) may be written as

$$P(n)_R = N \exp\{-2y_1^2(0)[\cos(\chi n t_1) - d_1(t_1)]^2\} P(n) , \tag{3.11}$$

where $d_1(t_1)$ is the inferred value of $\hat{D}_1(t_1)$ and N is a normalization constant.

If we make $y_1(0)$ large, i.e., the detector is prepared in a highly excited coherent state, then $P(n)_R = 0$ except when $d_1(t_1) = \cos(\chi n t_1)$. We conclude that $P(n)_R$ will have peaks at values of n given by

$$n = \frac{2\pi l}{\chi t_1} \pm \frac{\cos^{-1}[d_1(t_1)]}{\chi t_1} \quad l = 0, 1, \ldots \tag{3.12}$$

(where n must be an integer). This is what would be expected from a measurement of $\hat{D}_1(t)$, since these values of n correspond to the eigenstates of $\hat{D}_1(t)$ with eigenvalue equal to the inferred result.

In Figs. 2(a) and 2(b) we have plotted Eq. (3.11) for $P(n)$ Poissonian, i.e., the detector initially in a coherent state. As expected the limit of an accurate measurement of $\hat{D}_1(t)$ is obtained in the limit of $y_1(0)$ large. After such a measurement the detector finds itself in a superposition of the degenerate eigenstates of $\hat{D}_1(t)$; of course, which eigenstates result depend on the time at which the measurement is made. If $y_1(0)$ is not sufficiently large a couple of measurements made after commensurate evolution times will place the detector in a eigenstate of $\hat{D}_1(t)$. This is seen in Figs. 3(a)—3(c), where we have plotted $P(n)$ after three consecutive measurements. One must of course reprepare the detector in the same state prior to each measurement. This may be done with the coupling either on or off. After a series of such measurements the value for $\hat{D}_1(t_1)$ settles down to a steady value. This is what is required of a QND measurement.

It should be noted, however, that an initially precise measurement of $\hat{D}_1(t_1)$ at time t_1 cannot be used to infer a perfectly precise value for $\hat{D}_1(t_2)$. The reason for this is as follows. The QND condition (1.1) simply says that the system can exist in a simultaneous eigenstate of the QND variables $\hat{D}_1(t_1)$ and $\hat{D}_1(t_2)$; however, a problem arises when the eigenstates are degenerate. Consider the degenerate eigenstates of $\hat{D}_1(t_1)$. These are number states with n given by Eq. (3.12). Each of these number states is also an eigenstate of $\hat{D}_1(t_2)$. However, as mentioned in Sec. III these eigenstates are not degenerate with respect to $\hat{D}_1(t_2)$, but depend on the value of l. Thus while a linear superposition of the degenerate eigenstates of $\hat{D}_1(t_1)$ is also an eigenstate of $\hat{D}_1(t_1)$ it is not an eigenstate of $\hat{D}_1(t_2)$. In the problem we have considered a measurement of $\hat{D}_1(t_1)$ places the detector in just such a linear superposition of eigenstates. Since $a^\dagger a$ is a constant of motion the detector remains in this state under free evolution and thus is not in an eigenstate of $\hat{D}_1(t_2)$. More simply, even though $[\Delta D_1(t_1)]^2 = 0$, $[\Delta \hat{D}_1(t_2)]^2 \neq 0$. Despite the fact that the interaction is back action evading, $D_1(t_1)$ is only a stroboscopic QND variable.[1]

Since $a^\dagger a$ is a constant of motion, the detector, once placed in an eigenstate of $\hat{D}_1(t_1)$ will remain there. This enables a determinate sequence of results for a sequence of measurements of $\hat{D}_1(t_1)$ to be obtained, given perhaps one or two preparatory measurements.

However, if a classical force acts on an oscillator in an initial number state the oscillator is driven into a double-peaked distribution of number states [see Eq. (2.26) of Ref. 1]. Such a state will not be an eigenstate of $\hat{D}_1(t_1)$. Thus if the force acts at any time during the measurement sequence of $\hat{D}_1(t_1)$ one would obtain a result different from that obtained from previous measurements of $\hat{D}_1(t_1)$ and conclude that a classical force had been detected.

It would not be possible, however, to reconstruct the time dependence of the force. This is not surprising as the number operator is not a QND variable in the presence of the force. In the nomenclature of Unruh[1] $\hat{D}_1(t_1)$ is a QNDR (QND readout) observable but not a QNDD (QND detection) observable.[7] [Caves in Ref. 1 refers to QNDD observables as QNDF (force) observables.]

IV. CONCLUSION

We have demonstrated that for a harmonic-oscillator detector one can make QND measurements of the observable $\hat{D}_1(t) = \cos(\chi a^\dagger a t)$ by coupling the detector to another oscillator (meter) via an interaction represented by the interaction Hamiltonian $H_I = h\chi a^\dagger a b^\dagger b$. One can determine a value for $\hat{D}_1(t)$ by measuring the real part of the meter amplitude. This measurement of $\hat{D}_1(t)$ becomes more precise as the initial amplitude of the meter is increased. In the absence of a classical driving force on the detector the results obtained for $\hat{D}_1(t)$ in a sequence of measurements are entirely predictable from an initial precise measurement.

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