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Quantum-mechanical model for continuous position measurements

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We present an idealized model for a sequence of position measurements, and we then take an appropriate limit in which the measurements become continuous. The measurements lead to fluctuations without systematic dissipation, and they rapidly destroy off-diagonal terms in the position basis; thus the pointer basis is position. A modification of the model incorporates systematic dissipation via a feedback mechanism; in the modified model there is no decay of off-diagonal coherence in the position basis.

I. INTRODUCTION

In recent papers[1,2] Caves has presented a path-integral formalism which allows one to treat position measurements that are distributed in time, i.e., measurements that provide information about the position of a system at more than one time. In the limit of instantaneous position measurements, the formalism reduces to the conventional language of quantum mechanics—a system quantum state which undergoes unitary evolution between measurements and which suffers a nonunitary change ("state reduction") at the instant of each measurement. In this paper we consider, in terms of a particular model, a sequence of n such instantaneous measurements of position separated by a time τ. The individual measurements while instantaneous are not perfectly accurate. Our intention is to determine the evolution of the measured system in the continuous limit $n \rightarrow \infty$ and $\tau \rightarrow 0$. In this way we are able to describe a continuous or "dynamic" measurement of position.

Barchielli and co-workers[3,4] have developed an elegant formal description of continuous measurements. Their description of continuous position measurements[3] is the same as the description that we develop here based on a particular model, and many of our results have already been given by them. By basing our description directly on a model, however, we hope to make clearer the physical interpretation of the results. We also discuss the connection between our results and recent work in the theory of measurement, which relates the emergence of classical properties to interaction with an environment.[5,6]

In the standard formalism of quantum mechanics the change in the state of a system produced by an instantaneous, precise measurement may be calculated by using projection operators. In the case of position measurements such an approach is inappropriate. Formally this is because there are no normalized position eigenstates. Physically it is because an arbitrarily precise measurement of position requires arbitrarily strong coupling to the system and an arbitrarily large amount of energy. Thus in the case of position (or momentum) measurements one must generalize the standard formalism to include imprecision in the measurements. Such a generalization is provided by the formalism of "operations" and "effects."[7,8] (A similar argument is made in Ref. 9.) In fact, as discussed in Ref. 8, such a generalization is suggested by the mathematical structure of the standard quantum formalism and is required for a complete description of all measurements, not just measurements of observables with a continuous spectrum such as position.

We adopt this more general formalism here. As the formalism of operations and effects is perhaps not familiar to many readers, we present in Sec. II a brief summary of the formalism as it applies to position measurements. A more rigorous and complete discussion may be found in Refs. 7 and 8.

In Sec. III we introduce our model for a sequence of measurements of position. For this purpose one needs a collection of measuring devices referred to as "meters." Each meter may be regarded as the first stage of a genuine macroscopic measuring device. The model leads to an operation which specifies how each measurement changes the state of the system and a corresponding effect which determines the statistics of the outcomes. We analyze in detail the changes in the state of a free particle during a sequence of position measurements.

In Sec. IV we use the model to develop a description of continuous measurements. We consider a sequence of n position measurements, made in the time interval $(0,t]$ and separated by a time τ. We then take the continuous limit—$n \rightarrow \infty$ and $\tau \rightarrow 0$ with $n\tau = t$—and determine the time evolution of the measured system when no account is taken of the measured results. In terms of the formalism of operations and effects we find the "nonselective" operation for such a continuous sequence of measurements.

We analyze the time evolution in three equivalent ways, which correspond to (i) a Schrödinger-picture description, (ii) a Heisenberg-picture description, and (iii) a path-integral description. Each of these approaches shows that for this class of position measurements, the evolution of the state of the system is equivalent to a Gaussian quantum-dynamical semigroup.[10] In the Schrödinger picture we find that the evolution is governed by a master equation, in the Heisenberg description we obtain a set of Langevin equations, and in the path-integral description we are led to a particular form of the Feynman-Vernon influence functional.[11,12]

We find that the system density operator becomes diagonal in the position representation on a short time scale, while the momentum undergoes a random walk. The model does not, however, lead to any systematic dissipation. We are thus led in Sec. V to modify the model by including "feedback forces" which depend on the results of preceding measurements. The feedback forces give rise naturally to a dissipative contribution.

II. OPERATIONS AND EFFECTS

A thorough discussion of operations and effects may be found in Refs. 7 and 8. Here we present a summary, in Dirac notation, of as much of this formalism as will be required in subsequent sections.

Consider an ensemble of systems (free particles, for example) represented by a density operator $\hat{\rho}$. Each system interacts with a measuring device designed to measure position. By recording the relative frequencies with which the interactions produce a result between x and $x+dx$, we may construct (in principle at least) a probability density $P(x)$ for the measurement. Indeed the standard interpretation of quantum mechanics requires that just such a construction be possible.

In the standard theory it is assumed that there is no limit to the precision of such measurements; i.e., it is always possible to construct arbitrarily precise devices for which the probability density is determined only by the state of the system $\hat{\rho}$ through

$$P_s(x)=\mathrm{tr}(\,|x\rangle\langle x\,|\hat{\rho})=\langle x\,|\hat{\rho}\,|\,x\rangle \;. \tag{2.1}$$

The probability density $P_s(x)$ can be arbitrarily narrow, the only restrictions imposed by quantum mechanics being positivity and normalizability. We have already recognized, however, that the construction of such arbitrarily precise measuring devices is at least very difficult if not physically impossible. In any case it seems unnecessarily restrictive to consider only such idealized instruments. Is it possible to include a more general class of measurements in the formalism of quantum mechanics?

To do this we postulate that for a particular measuring device (for position), labeled with a parameter σ, the resulting densities $P_\sigma(x)$, which depend on the properties of the device as well as the state of the system, are restricted to a "minimum width" subset of the possible probability densities $P(x)$. This may be represented formally by writing

$$\{P_\sigma(x)\}=\{P(x)\,|\,0\le P(x)\le H_\sigma\} \;, \tag{2.2}$$

where H_σ is a constant depending on σ. That such a condition restricts the width of $P_\sigma(x)$ may be seen by noting that the area under $P_\sigma(x)$ is determined by normalization to be unity. The more one tries to "confine" $P_\sigma(x)$ to small intervals of the real line, the higher it must grow. Thus placing a restriction on this growth restricts the width of $P_\sigma(x)$.

We are thus led to introduce bounded positive (self-adjoint) operators $\hat{F}_\sigma(x)$ which determine $P_\sigma(x)$ by

$$P_\sigma(x)=\mathrm{tr}[\hat{F}_\sigma(x)\hat{\rho}] \;. \tag{2.3}$$

Equation (2.2) requires that

$$\hat{0}\le\hat{F}_\sigma(x)\le H_\sigma\hat{1} \;. \tag{2.4}$$

Normalization of the probability density $P_\sigma(x)$ for arbitrary $\hat{\rho}$ further requires that

$$\int_{-\infty}^{\infty} dx\,\hat{F}_\sigma(x)=\hat{1} \;. \tag{2.5}$$

To ensure that this generalization corresponds to a position measurement we also place the condition

$$[\hat{x},\hat{F}_\sigma(x)]=0 \;. \tag{2.6}$$

Hence $\hat{F}_\sigma(x)$ is diagonal in the position representation,

$$\langle x'\,|\,\hat{F}_\sigma(x)\,|\,x''\rangle=\mathscr{f}_\sigma(x,x')\delta(x'-x'') \;. \tag{2.7}$$

Equation (2.3) leads to a simple interpretation of $\mathscr{f}_\sigma(x,x')$. One easily shows that

$$P_\sigma(x)=\int_{-\infty}^{\infty} dx'\,\mathscr{f}_\sigma(x,x')P_s(x') \;, \tag{2.8}$$

where $P_s(x)$ is given by Eq. (2.1). We may thus interpret $\mathscr{f}_\sigma(x,x')$ as a conditional probability density, normalized by Eq. (2.5).

We may regard $\hat{F}_\sigma(x)dx$ as an "effect-valued measure," which generalizes the "projection-valued measures" of the standard formalism. We shall refer to $\hat{F}_\sigma(x)$ as an "effect density." The effect density incorporates the properties of the measuring device which affect $P_\sigma(x)$.

As an example (which arises naturally in Sec. III A) consider[3]

$$\hat{F}_\sigma(x)=(\pi\sigma)^{-1/2}\exp[-(x-\hat{x})^2/\sigma] \;. \tag{2.9}$$

One easily verifies that $\hat{F}_\sigma(x)$ is an effect density for a position measurement; i.e., it satisfies Eqs. (2.4)–(2.6) with $H_\sigma=(\pi\sigma)^{-1/2}$. Furthermore, writing

$$\hat{F}_\sigma(x)=(\pi\sigma)^{-1/2}\times\int_{-\infty}^{\infty} dx'\,\exp[-(x-x')^2/\sigma]\,|\,x'\rangle\langle x'\,| \;,$$

one sees that

$$\lim_{\sigma\to 0}\hat{F}_\sigma(x)=|\,x\rangle\langle x\,| \;;$$

thus the standard projection-valued measure is recovered in the limit.

The effect density introduced above determines the statistics of the outcomes of the measurement. We would also like to know how the measurement changes the state of the measured system. This requires the concept of an operation.

Let there be N elements in the original ensemble described by a density operator $\hat{\rho}$. The number of systems for which a measurement of position gives a result lying in some interval Δ is given by

$$N_\Delta = N \int_\Delta dx\ \mathrm{tr}[\hat{F}_\sigma(x)\hat{\rho}] \ . \tag{2.10}$$

After interaction with the measuring device we select these N_Δ systems and form a new ensemble to be described by a density operator $\hat{\rho}_\Delta$. An operation is a linear mapping Φ_Δ (on the space of trace-class operators) which determines $\hat{\rho}_\Delta$ by

$$\Phi_\Delta \hat{\rho} = \lim_{N\to\infty} (N_\Delta/N)\hat{\rho}_\Delta \ . \tag{2.11}$$

Thus the probability that a measurement gives a result in the interval Δ is

$$P(x\in\Delta) = \mathrm{tr}(\Phi_\Delta\hat{\rho}) = \int_\Delta dx\ \mathrm{tr}[\hat{F}_\sigma(x)\hat{\rho}] \ . \tag{2.12}$$

We can derive the operation Φ_Δ from an "operation density" ϕ_x,

$$\Phi_\Delta = \int_\Delta dx\ \phi_x \ . \tag{2.13}$$

The operation density ϕ_x in turn determines the effect density $\hat{F}_\sigma(x)$ through the relation

$$\mathrm{tr}(\phi_x\hat{\rho}) = \mathrm{tr}[\hat{F}_\sigma(x)\hat{\rho}] = P_\sigma(x) \ . \tag{2.14}$$

As ϕ_x determines $\hat{F}_\sigma(x)$ only through the diagonal elements of $\phi_x\hat{\rho}$, there are many ϕ_x corresponding to a particular $\hat{F}_\sigma(x)$. This simply means that there is more than one measuring instrument corresponding to a particular class $\{P_\sigma(x)\}$ in Eq. (2.2). Specification of ϕ_x defines a particular instrument.

The above definition of the operation Φ_Δ corresponds to a "selective operation;" i.e., it determines a new density operator for the system when we take account of the measured result. We may also define a nonselective operation

$$\Phi \equiv \int_{-\infty}^{\infty} dx\ \phi_x \tag{2.15}$$

by selecting all elements of the original ensemble to form the post-measurement ensemble, regardless of the measured result. Such an operation tells us how the system's state changes when no account is taken of the measured result.

While $\mathrm{tr}(\Phi\hat{\rho}) = \int_{-\infty}^{\infty} dx\ P_\sigma(x) = 1$, it can be shown that in general $\mathrm{tr}[(\Phi\hat{\rho})^2] \neq 1$. Thus a nonselective operation in general takes pure states to mixed states—not a surprising result since Φ describes the situation where one discards all the information acquired in the measurement. The nonselective operation Φ describes the irreversible interaction of the system with the measuring device.

As an example consider the operation density defined by

$$\phi_x\hat{\rho} = (\pi\sigma)^{-1/2} \exp[-(x-\hat{x})^2/2\sigma + ik\hat{x}]\hat{\rho} \times \exp[-(x-\hat{x})^2/2\sigma - ik\hat{x}] \ . \tag{2.16}$$

Via Eq. (2.14) this operation density determines the effect density (2.9). The corresponding nonselective operation is

$$\Phi\hat{\rho} = (\pi\sigma)^{-1/2} \int_{-\infty}^{\infty} dx\ \exp[-(x-\hat{x})^2/2\sigma + ik\hat{x}]\hat{\rho} \times \exp[-(x-\hat{x})^2/2\sigma - ik\hat{x}] \ . \tag{2.17}$$

III. POSITION MEASUREMENTS

A. Model for position measurements

We now want to model a sequence of instantaneous position measurements at times $t_r = r\tau$ $(r=1,\ldots,n)$. We use a model that goes back to von Neumann.[13] The system is coupled to a sequence of measuring apparatuses called meters. The meter for the rth measurement has canonical variables $\hat{\bar{x}}_r$ and $\hat{\bar{p}}_r$. Each of the meters can be regarded as the first stage of a macroscopic measuring apparatus. For convenience we assume that the free Hamiltonian of the meters is proportional to the identity and thus can be ignored. The total Hamiltonian for the system and the meters is taken to be[1]

$$\hat{H} = \hat{H}_0 + \sum_{r=1}^{n} \delta(t - r\tau)\hat{x}\hat{\bar{p}}_r \ , \tag{3.1}$$

where $\hat{H}_0$ is the free Hamiltonian of the system.

It should be emphasized that the Hamiltonian (3.1) depends explicitly on time, reflecting the fact that together the system and the meters do not constitute a closed system. They are open to subsequent stages of the measuring apparatus, which determine the time dependence (for a discussion of this point see Ref. 14). We assume a δ-function time dependence, which corresponds to an ideal measuring apparatus making instantaneous measurements. A more general theory of measurement dynamics would relax this assumption.[1,2] We simply note here that the form of the time dependence in Eq. (3.1) is an essential part of the definition of the measurement process.

Assume now that all measurements in the sequence are described in the same way, so that we may focus attention on a representative measurement, the rth. Assume further that the rth meter is prepared in a pure state $|\Upsilon_r\rangle$ with Gaussian wave function

$$\langle \bar{x}_r | \Upsilon_r \rangle = \Upsilon(\bar{x}_r) = (\pi\sigma)^{-1/4} \exp(-\bar{x}_r^2/2\sigma) \ . \tag{3.2}$$

Let $\hat{\rho}_r(t_r-)$ be the state of the system just prior to the rth measurement at time $t_r = r\tau$; this state depends implicitly on the results of the preceding measurements. Immediately after the rth δ-function interaction in Eq. (3.1) the joint state of the system and the rth meter is given by

$$\hat{R}^{(r)} = e^{-i\hat{x}\hat{\bar{p}}_r/\hbar}[\,|\Upsilon_r\rangle\langle\Upsilon_r| \otimes \hat{\rho}_r(t_r-)]e^{i\hat{x}\hat{\bar{p}}_r/\hbar} \ . \tag{3.3}$$

At the same instant we assume that subsequent stages of the measuring apparatus make an arbitrarily precise measurement of the rth meter's coordinate, the result being $\bar{x}_r$. As we have already noted such a situation is difficult to realize, but it should provide a good approximation for determining the change of the system state because it includes imprecision due to the quantum mechanics of the meter. One must put the quantum-to-classical cut somewhere, and we choose to put it just beyond the meter.

Define now a system operator

$$\hat{\Upsilon}(\bar{x}_r) \equiv \langle \bar{x}_r | e^{-i\hat{x}\hat{p}_r/\hbar} | \Upsilon_r \rangle \; , \tag{3.4}$$

and notice that it can be written in terms of the wave function (3.2) as

$$\hat{\Upsilon}(\bar{x}_r) = \Upsilon(\bar{x}_r - \hat{x}) = (\pi\sigma)^{-1/4} \exp[-(\bar{x}_r - \hat{x})^2/2\sigma] \; . \tag{3.5}$$

Using Eqs. (3.4) and (3.5) one can show that the probability density to obtain $\bar{x}_r$ as the result of the rth measurement is given by

$$P(\bar{x}_r) = \mathrm{tr}(\, |\bar{x}_r\rangle\langle\bar{x}_r| \hat{R}^{(r)}) = \mathrm{tr}[\hat{F}_\sigma(\bar{x}_r)\hat{\rho}_r(t_r -)] \; , \tag{3.6}$$

where

$$\hat{F}_\sigma(\bar{x}_r) \equiv \hat{\Upsilon}^\dagger(\bar{x}_r)\hat{\Upsilon}(\bar{x}_r) = (\pi\sigma)^{-1/2} \exp[-(\bar{x}_r - \hat{x})^2/\sigma] \tag{3.7}$$

is an effect density [cf. Eq. (2.9)].

Given the result $\bar{x}_r$, the state of the system just after the rth measurement is

$$\hat{\rho}_{r+1}(t_r +) = \frac{\langle \bar{x}_r | \hat{R}^{(r)} | \bar{x}_r \rangle}{P(\bar{x}_r)} = \frac{\phi_{\bar{x}_r}\hat{\rho}_r(t_r -)}{P(\bar{x}_r)} \; . \tag{3.8}$$

The operation density $\phi_{\bar{x}_r}$ is defined by

$$\phi_{\bar{x}_r}\hat{\rho} \equiv \hat{\Upsilon}(\bar{x}_r)\hat{\rho}\hat{\Upsilon}^\dagger(\bar{x}_r) \tag{3.9}$$

[cf. Eq. (2.16) with $k=0$]. The corresponding nonselective operation is given by

$$\Phi\hat{\rho} = \int_{-\infty}^{\infty} d\bar{x}_r \, \hat{\Upsilon}(\bar{x}_r)\hat{\rho}\hat{\Upsilon}^\dagger(\bar{x}_r) \; . \tag{3.10}$$

Equations (3.4)–(3.10) show how operations and effects arise naturally in this idealized model of a position measurement. It should be emphasized that $\phi_{\bar{x}_r}$ and $\hat{F}_\sigma(\bar{x}_r)$ are labeled by the position variable of the meter, $\bar{x}_r$; hence $\phi_{\bar{x}_r}$ and $\hat{F}_\sigma(\bar{x}_r)$ are written directly in terms of results of measurements—not in terms of system variables. This is a general feature of a description involving operations and effects.

Barchielli, Lanz, and Prosperi[3] use the operation (3.10) and the effect (3.7) as the basis for the development of their formal description of continuous position measurements. We show here how this operation and effect arise from a measurement model that uses standard quantum mechanics (see also Ref. 1). Thus we are able to contradict an assertion by d'Espagnat,[15] who contends that this description of a position measurement is inconsistent with standard quantum mechanics (for further discussion see Ref. 16).

Return now to the sequence of n position measurements separated by a time τ. Between any two measurements the system evolves freely via the unitary transformation

$$\hat{U}(\tau) = e^{-i\hat{H}_0\tau/\hbar} \; . \tag{3.11}$$

Thus if $\hat{\rho}(0)$ is the initial state of the system at $t=0$ and if the results of the measurements form the sequence $\bar{x}_1, \ldots, \bar{x}_n \equiv \{\bar{x}_r\}$, the state of the system just after the nth measurement (time $t_n = n\tau$) is[1(b)]

$$\hat{\rho}(\{\bar{x}_r\}, t_n +) \equiv \hat{\rho}_{n+1}(t_n +) = \frac{\left[\prod_{r=1}^{n} \hat{\Upsilon}(\bar{x}_r)\hat{U}(\tau)\right] \hat{\rho}(0) \left[\prod_{r=1}^{n} \hat{\Upsilon}(\bar{x}_r)\hat{U}(\tau)\right]^\dagger}{P(\{\bar{x}_r\})} \; , \tag{3.12}$$

where

$$P(\{\bar{x}_r\}) \equiv \mathrm{tr}\left[\left[\prod_{r=1}^{n} \hat{\Upsilon}(\bar{x}_r)\hat{U}(\tau)\right]^\dagger \left[\prod_{r=1}^{n} \hat{\Upsilon}(\bar{x}_r)\hat{U}(\tau)\right] \hat{\rho}(0) \right] \tag{3.13}$$

is the joint probability density to obtain the sequence of results $\{\bar{x}_r\}$. In Eqs. (3.12) and (3.13) the products are ordered with increasing values of r on the left. Equation (3.12) gives the Schrödinger-picture evolution of the system with results $\{\bar{x}_r\}$ selected. The corresponding nonselective evolution is given by

$$\hat{\rho}(t_n +) = \int \left[\prod_{r=1}^{n} d\bar{x}_r\right] \hat{\rho}(\{\bar{x}_r\}, t_n +) P(\{\bar{x}_r\}) \; . \tag{3.14}$$

B. Selective evolution of a free particle

In order to clarify the preceding general description we consider in detail the *selective* evolution of a free particle with mass m, position $\hat{x}$, momentum $\hat{p}$, and free Hamiltonian $\hat{H}_0 = \hat{p}^2/2m$. Let $\hat{\rho}_r(t)$ denote the state of the system during the interval between the $(r-1)$th and rth measurements $(t_{r-1} < t < t_r)$. Both unitary Schrödinger evolution (3.11) and the operation density (3.9) take Gaussian pure states to Gaussian pure states.

Thus it is consistent to assume that $\hat{\rho}_r(t) = |\psi_r(t)\rangle\langle\psi_r(t)|$, where $|\psi_r(t)\rangle$ is a pure state with Gaussian wave function[17]

$$\psi_r(x,t)=e^{i\phi_r(t)}[\pi\Delta_r(t)]^{-1/4} \times\exp\left[-\frac{1-i\epsilon_r(t)}{2\Delta_r(t)}[x-a_r(t)]^2+\frac{i}{\hbar}b_r(t)x\right] . \tag{3.15}$$

The parameters a_r, b_r, Δ_r, ϵ_r, and ϕ_r are all functions of time, which vary smoothly under unitary evolution during the interval (t_{r-1},t_r). They determine the expectation values and second moments for the wave packet (3.15):

$$\langle\hat{x}\rangle=a_r(t) , \tag{3.16a}$$

$$\langle\hat{p}\rangle=b_r(t) , \tag{3.16b}$$

$$\langle(\Delta\hat{x})^2\rangle=\tfrac{1}{2}\Delta_r(t) , \tag{3.17a}$$

$$\langle(\Delta\hat{p})^2\rangle=\frac{1+\epsilon_r^2(t)}{\Delta_r(t)}\frac{\hbar^2}{2} , \tag{3.17b}$$

$$\langle\Delta\hat{x}\,\Delta\hat{p}+\Delta\hat{p}\,\Delta\hat{x}\rangle=\hbar\epsilon_r(t) . \tag{3.17c}$$

The parameters a_r and b_r—first-moment parameters—give directly the position and momentum expectation values, whereas the parameters Δ_r and ϵ_r—second-moment parameters—determine the width of the wave packet and the correlation between position and momentum. The phase ϕ_r has no effect on measurable quantities so we neglect it in what follows.

We are interested in the values of the parameters just after the $(r-1)$th measurement and just before the rth measurement. Throughout the following we adopt a shorthand notation: a parameter with no adornment denotes a value just after the $(r-1)$th measurement [e.g., $\Delta_r\equiv\Delta_r(t_{r-1}+)$], and a parameter with a prime denotes a value just before the rth measurement [e.g., $\Delta_r'\equiv\Delta_r(t_r-)$]. Unitary evolution relates the primed parameters to the unprimed parameters,

$$a_r'=a_r+b_r\tau/m , \tag{3.18a}$$

$$b_r'=b_r , \tag{3.18b}$$

$$\Delta_r'=\Delta_r+\frac{1+\epsilon_r^2}{\Delta_r}\left[\frac{\hbar\tau}{m}\right]^2+2\epsilon_r\frac{\hbar\tau}{m} , \tag{3.19a}$$

$$\epsilon_r'=\epsilon_r+\frac{1+\epsilon_r^2}{\Delta_r}\frac{\hbar\tau}{m} . \tag{3.19b}$$

The statistics of the rth measurement are determined by the primed parameters; the probability density $P(\bar{x}_r)$ [Eq. (3.6)] is a Gaussian with mean and variance

$$\langle\bar{x}_r\rangle=a_r' , \tag{3.20}$$

$$\langle(\Delta\bar{x}_r)^2\rangle=\tfrac{1}{2}(\Delta_r'+\sigma) . \tag{3.21}$$

As a consequence of the rth measurement with result $\bar{x}_r$, the system state changes discontinuously according to Eq. (3.8). The system state just after the rth measurement is a pure state

$$|\psi_{r+1}(t_r+)\rangle=\hat{\Upsilon}(\bar{x}_r)\,|\psi_r(t_r-)\rangle/[P(\bar{x}_r)]^{1/2} , \tag{3.22}$$

whose Gaussian wave function $\psi_{r+1}(x,t_r+) \propto\Upsilon(\bar{x}_r-x)\psi_r(x,t_r-)$ is specified (aside from a phase) by the parameters[18]

$$a_{r+1}=a_r'+\frac{C_r-1}{C_r}(\bar{x}_r-a_r') , \tag{3.23a}$$

$$b_{r+1}=b_r'+\hbar\frac{\epsilon_r'}{\Delta_r'}\frac{C_r-1}{C_r}(\bar{x}_r-a_r') , \tag{3.23b}$$

$$\Delta_{r+1}=\Delta_r'/C_r , \tag{3.24a}$$

$$\epsilon_{r+1}=\epsilon_r'/C_r . \tag{3.24b}$$

In these equations

$$C_r\equiv1+\Delta_r'/\sigma\geq1 \tag{3.25}$$

is a "contraction factor"—i.e., the factor by which the position variance decreases as a consequence of the rth measurement. Notice that if $\Delta_r'\gg\sigma$ then $C_r\simeq\Delta_r'/\sigma$, so that one measurement suffices to reduce the position variance to $\simeq\sigma/2$.

The second-moment parameters change in a completely predictable way throughout the sequence [Eqs. (3.19 and (3.24)]; i.e., their changes at a measurement do not depend on the result of the measurement. In contrast the first-moment parameters "jump" at each measurement in a way that depends on the result of the measurement [Eqs. (3.23)]. Thus the expected result of a particular measurement [Eq. (3.20)] depends on the results of all preceding measurements.

Consider now a long sequence of measurements characterized by a particular value of σ. Since the second-moment parameters change predictably, one may seek a stationary configuration in which the second-moment parameters assume the same values Δ and ϵ just *after* each measurement. Let Δ' and ϵ' denote the corresponding stationary values just *before* each measurement. Rather than regarding σ as fixed, it is simpler and equivalent to seek a stationary configuration for a given contraction factor $C=1+\Delta'/\sigma$. Thus the conditions for stationarity become

$$C\Delta=\Delta+\frac{1+\epsilon^2}{\Delta}\left[\frac{\hbar\tau}{m}\right]^2+2\epsilon\frac{\hbar\tau}{m} , \tag{3.26a}$$

$$C\epsilon=\epsilon+\frac{1+\epsilon^2}{\Delta}\frac{\hbar\tau}{m} \tag{3.26b}$$

[Eqs. (3.19) and (3.24)], which one solves for Δ and ϵ as functions of C,

$$\Delta=\frac{\Delta'}{C}=C^{-1/2}\frac{C+1}{C-1}\frac{\hbar\tau}{m} , \tag{3.27a}$$

$$\epsilon=\frac{\epsilon'}{C}=C^{-1/2} . \tag{3.27b}$$

The corresponding value of σ is

$$\sigma=\frac{\Delta'}{C-1}=C^{1/2}\frac{C+1}{(C-1)^2}\frac{\hbar\tau}{m} . \tag{3.28}$$

The stationary variance of the measurements [Eq. (3.21)], given by

$$\langle(\Delta\bar{x}_r)^2\rangle=\tfrac{1}{2}(\Delta'+\sigma)=\tfrac{1}{2}C\sigma=C^{3/2}\frac{C+1}{(C-1)^2}\frac{\hbar\tau}{2m} , \tag{3.29}$$

has a minimum value $\simeq 2.05(\hbar\tau/m)$ for $C=3+2\sqrt{3} \simeq 6.46$ [$\sigma \simeq 0.64(\hbar\tau/m)$].

A tedious linear-stability analysis, which we sketch in Appendix A, shows that the stationary solution (3.27) is stable under small perturbations. Perhaps all sequences, with arbitrary (Gaussian) initial conditions, ultimately approach the stationary configuration. Once a sequence has settled into the stationary configuration each measurement reduces the value of the position variance from $C\Delta/2$ just before the measurement to $\Delta/2$ just after; wave-packet spreading then increases the position variance back to $C\Delta/2$ just before the next measurement. Similar considerations hold for the correlation parameter ϵ.

We turn now to the behavior of the mean position and mean momentum. In the stationary configuration the mean position and mean momentum change according to

$$a_{r+1}-a_r=b_r\frac{\tau}{m}+\frac{C-1}{C}(\bar{x}_r-a_r') , \tag{3.30a}$$

$$b_{r+1}-b_r=\frac{m}{\tau}\frac{(C-1)^2}{C(C+1)}(\bar{x}_r-a_r')=\frac{\hbar}{C^{1/2}\sigma}(\bar{x}_r-a_r') \tag{3.30b}$$

[Eqs. (3.18), (3.23), (3.27), and (3.28)]. We concentrate here on the behavior of the mean momentum. The quantities $\bar{x}_r-a_r'$ may be regarded as uncorrelated, zero-mean, Gaussian random variables with stationary variance $C\sigma/2$ [Eq. (3.29)]. Thus the momentum jump (3.30b) means that the mean momentum undergoes a random walk with step size $(\hbar/C^{1/2}\sigma)(C\sigma/2)^{1/2}=(\hbar^2/2\sigma)^{1/2}$. This step size may be readily understood in terms of a "back-action disturbance" from the meter. The Hamiltonian (3.1) implies that at each measurement the system momentum receives a "back-action kick" equal to the meter momentum. The size of this back-action kick is characterized by the uncertainty in the meter momentum; for the meter wave function (3.2) that uncertainty yields the step size $(\hbar^2/2\sigma)^{1/2}$.

Using the same measurement model, Lamb[18] has analyzed a sequence of position measurements on a free particle. He simulates the selective evolution on a computer in order to investigate how the expectation value of the particle's position changes during a sequence of measurements.

During a time t in which there t/τ measurements the mean momentum diffuses by an amount $(\hbar^2/2\sigma)^{1/2}(t/\tau)^{1/2}=(\hbar^2/2D)^{1/2}t^{1/2}$, where $D=\sigma\tau$. There is an important lesson here: one may increase the momentum disturbance either by decreasing σ (more "accurate" meters) or by decreasing τ (more measurements), but all that matters for the momentum disturbance during a given time is the product $D=\sigma\tau$. In Sec. IV we wish to take the continuous limit $\tau\to 0$; these considerations suggest that we must simultaneously take the limit $\sigma\to\infty$ with $D=\sigma\tau$ held constant.[3] This procedure for taking the continuous limit is calculated to keep constant the momentum disturbance within a given time. Although this conclusion emerges here from analysis of a very special case—measurements on a free particle—one might expect that for a general quantum system the momentum diffusion is superposed on the intrinsic dynamics of the system. This expectation is confirmed in Sec. IV.

Notice that in the continuous limit ($\tau\to 0$, $\sigma\to\infty$, with $D=\sigma\tau$ held constant) the contraction factor limits to 1. Indeed Eq. (3.28) shows that

$$\frac{\tau}{C-1}\to\left[\frac{m}{2\hbar}\right]^{1/2}D^{1/2}\equiv t_c . \tag{3.31a}$$

Thus the second-moment parameters (3.27) have continuous limits

$$\Delta\to\Delta_c\equiv\left[\frac{2\hbar}{m}\right]^{1/2}D^{1/2} , \tag{3.31b}$$

$$\epsilon\to\epsilon_c=1 . \tag{3.31c}$$

The linear-stability analysis in Appendix A reveals that the e-folding time for approaching the stationary configuration in the continuous limit is the time t_c defined in Eq. (3.31a). Notice that $\Delta_c t_c=D$. The significance of t_c is also apparent in the ratio of the stationary momentum variance to the diffusion of the mean momentum,

$$\frac{\langle(\Delta\hat{p})^2\rangle}{(\hbar^2/2D)t}=\frac{(1+\epsilon_c^2)/\Delta_c}{t/D}=2\frac{t_c}{t} . \tag{3.32}$$

The limiting quantities (3.31) appear in the analysis of Appendix B.

One qualitative conclusion may be drawn from the momentum diffusion: during a sequence of position measurements the mean momentum of a free particle wanders away from its initial value without bound. This is due to the lack of any systematic dissipation in the model. It is clearly an unrealistic feature, because in a real experiment there would be some feedback mechanism to bound the momentum diffusion. In Sec. V we show how to include in the model "feedback forces" which lead to a systematic dissipation. The limiting quantities (3.31) play an important role in the feedback analysis.

Before going on to the nonselective evolution, we draw attention to a realization of a continuous measurement—a continuous measurement of the spin of an electron, the theory for which has been discussed by Dehmelt.[19]

IV. CONTINUOUS POSITION MEASUREMENTS

We now proceed to analyze the *nonselective* evolution of the system in the continuous limit $n\to\infty$, $\tau\to 0$, with $t_n=n\tau=t$. The analysis in Sec. III B suggests that we must simultaneously take the limit $\sigma\to\infty$ with $D=\sigma\tau$

held constant. This limit has been formulated previously by Barchielli, Lanz, and Prosperi,[3] and it leads to their formal description of continuous position measurements. Ghirardi, Rimini, and Weber[20] have presented a similar formal description for the case where the individual position measurements occur randomly in time.

We perform the continuous limit in three equivalent pictures.

A. Schrödinger picture: Master equation

Our objective is to derive an evolution equation—a master equation—for the system density operator $\hat{\rho}(t)$ when no account is taken of the measured results. To that end define the time derivative of $\hat{\rho}(t)$ by

$$\frac{d\hat{\rho}(t)}{dt} \equiv \lim_{\tau\to 0} \frac{\hat{\rho}(t_n+)-\hat{\rho}(t_{n-1}+)}{\tau} , \tag{4.1}$$

where $\hat{\rho}(t_n+)$ and $\hat{\rho}(t_{n-1}+)$ are the nonselective density operators just after the nth and $(n-1)$th measurements [Eq. (3.14)]. Using Eqs. (3.12) and (3.14) we may write the time derivative as

$$\frac{d\hat{\rho}(t)}{dt} = \lim_{\tau\to 0}\left[\frac{1}{\tau}\int_{-\infty}^{\infty} d\bar{x}\, \hat{\Upsilon}(\bar{x})\hat{U}(\tau)\hat{\rho}(t_{n-1}+)\hat{U}^{\dagger}(\tau)\hat{\Upsilon}^{\dagger}(\bar{x}) - \frac{1}{\tau}\hat{\rho}(t_{n-1}+)\right] . \tag{4.2}$$

To simplify this expression we may use the results of Gaussian integrals over $\bar{x}^n$ to show that for any operator $\hat{A}$

$$\int_{-\infty}^{\infty} d\bar{x}\, \hat{\Upsilon}(\bar{x})\hat{A}\hat{\Upsilon}^{\dagger}(\bar{x}) = \hat{A} - \frac{1}{4\sigma}[\hat{x},[\hat{x},\hat{A}]] + O\left[\frac{1}{\sigma^2}\right] , \tag{4.3}$$

where $O(1/\sigma^2)$ indicates terms of order higher than one in $1/\sigma$. In the limit of small τ we may also use

$$\hat{U}(\tau) \simeq \hat{1} - i\hat{H}_0\tau/\hbar . \tag{4.4}$$

Substituting Eqs. (4.3) and (4.4) into Eq. (4.2) and taking the continuous limit ($\tau\to 0$, $\sigma\to\infty$, with $D=\sigma\tau=\text{const}$), we find the master equation[3]

$$\frac{d\hat{\rho}(t)}{dt} = -\frac{i}{\hbar}[\hat{H}_0,\hat{\rho}(t)] - \frac{1}{4D}[\hat{x},[\hat{x},\hat{\rho}(t)]] . \tag{4.5}$$

The form of Eq. (4.5) shows that during a continuous sequence of position measurements the system evolves through the action of a Gaussian quantum-dynamical semigroup.[10] The constant D characterizes a class of "dynamic" or continuous position measurements and must be taken as given in the specification of the measuring apparatus. Of course the master equation (4.5) is really an approximation to a realistic situation in which τ is small but nonzero. We defer discussion of this point until the end of Sec. III A.

The first term in Eq. (4.5) describes the intrinsic dynamics of the system under the action of its free Hamiltonian $\hat{H}_0$. The second term arises from the coupling to the meters; it describes a quantum Wiener process (diffusion) in momentum, for which there is no associated dissipation—fluctuations without dissipation.

As an example consider a free particle. The absence of dissipation is apparent from the equations of motion for the mean position and mean momentum,

$$\frac{d\langle\hat{x}\rangle}{dt} = \frac{\langle\hat{p}\rangle}{m} , \tag{4.6a}$$

$$\frac{d\langle\hat{p}\rangle}{dt} = 0 , \tag{4.6b}$$

which are just those of the intrinsic dynamics. The Wiener process in momentum shows up in the momentum variance

$$\langle(\Delta\hat{p})^2\rangle = \langle(\Delta\hat{p})^2\rangle_0 + \hbar^2 t/2D , \tag{4.7}$$

which displays diffusion with diffusion constant $\hbar^2/2D$. This diffusion of the momentum variance in the nonselective evolution is the exact counterpart of the diffusion of the *mean* momentum in the selective evolution discussed in Sec. III B. For a general quantum system the equations of motion for the mean position and mean momentum are still those of the intrinsic dynamics, with no evidence of dissipation, and the momentum diffusion is superposed on the intrinsic dynamics.

An important consequence of Eq. (4.5) shows up in the matrix elements of $\hat{\rho}(t)$ in the position representation,

$$\rho(x,x',t) \equiv \langle x\,|\,\hat{\rho}(t)\,|\,x'\rangle . \tag{4.8}$$

Using Eq. (4.5) one finds that

$$\frac{\partial\rho(x,x',t)}{\partial t} = (\text{intrinsic dynamics}) - \frac{1}{4D}(x-x')^2\rho(x,x',t) , \tag{4.9}$$

where "intrinsic dynamics" denotes the contribution arising from the first term in Eq. (4.5). Equation (4.9) suggests that off-diagonal elements in the position representation decay at a rate determined by a constant $\gamma \equiv 1/4D$ times the square of the separation. In Appendix B we investigate this suggestion for a free particle. Such evolution has received extensive treatment by Zurek[5] and by Joos and Zeh[6] in the context of quantum measurement theory, and it has been investigated in other contexts as well.[21–24] In the language of Zurek this evolution establishes a "pointer basis," the representation in which $\hat{\rho}(t)$ tends to become diagonal. Not surprisingly the pointer basis in this case turns out to be position. In physical terms the decay of off-diagonal coherence means that quantum interference effects across a distance L are destroyed on a time scale $\sim(\gamma L^2)^{-1} = 4D/L^2$.

Once again we see the crucial role played by the parameter D: the smaller the value of D, the more rapid is the decay of off-diagonal coherence. What determines the size of D? Real position measurements provide information with some accuracy $(\sigma/2)^{1/2}$ over a bandwidth $B \simeq 1/2\tau$. The parameter D is then determined to be $D=\sigma\tau\simeq\sigma/2B$. The important question becomes the following: when is it permissible to replace the real measurement process, which has finite accuracy and finite bandwidth, by the corresponding continuous limit, in which both σ and B go to infinity? The answer is a standard one. The continuous limit should provide a good description of a real measurement process with the same value of D, so long as all times of interest are somewhat longer than $\tau \simeq 1/2B$.

B. Heisenberg picture: Langevin equation

In Sec. IV A we obtained a master equation for $\hat{\rho}(t)$ in the case of nonselective position measurements. In this section our objective is to find equivalent equations of motion for the system operators $\hat{x}(t)$ and $\hat{p}(t)$; i.e., we wish to determine the evolution in the Heisenberg picture.

The following Heisenberg equations for the system operators may be obtained directly from the Hamiltonian (3.1):

$$\frac{d\hat{x}}{dt} = -\frac{i}{\hbar}[\hat{x},\hat{H}_0] , \tag{4.10a}$$

$$\frac{d\hat{p}}{dt} = -\frac{i}{\hbar}[\hat{p},\hat{H}_0]+\hat{F}_p(t) . \tag{4.10b}$$

Equation (4.10b) has the form of a quantum Langevin equation for momentum,[25] with the Langevin force given by

$$\hat{F}_p(t) \equiv -\sum_{r=1}^{n} \delta(t-r\tau)\hat{\bar{p}}_r . \tag{4.11}$$

The meter momenta $\hat{\bar{p}}_r$ in this expression are constants of the motion. To solve Eqs. (4.10) one needs to know the mean and the two-time correlation function for $\hat{F}_p(t)$.

One easily shows, using Eq. (3.2) for the state of each meter, that

$$\langle \hat{F}_p(t) \rangle = 0 . \tag{4.12}$$

With a bit more effort one may show (Appendix C) that in the continuous limit the two-time correlation function is

$$\langle \hat{F}_p(t)\hat{F}_p(t') \rangle = \frac{\hbar^2}{2D}\delta(t-t') . \tag{4.13}$$

Equation (4.13) confirms our previous identification of a Wiener process in momentum with diffusion constant $\hbar^2/2D$. That the Langevin force is δ correlated is a consequence of the continuous limit. In a real measurement process the Langevin force would have a finite bandwidth B. As discussed in Sec. IV A the continuous limit should give a good description provided all times of interest are somewhat longer than $1/2B$.

C. Path-integral picture: Influence functional

The nonselective evolution of $\hat{\rho}(t)$ may also be represented by a path integral. Recall that $\hat{\rho}(t_n+)$ [Eq. (3.14)] denotes the nonselective density operator for the system just after the completion of n measurements at times $t_r = r\tau$, $r=1,\ldots,n$. Let $\rho(x,x',t_n+) \equiv \langle x \mid \hat{\rho}(t_n+) \mid x' \rangle$ be the corresponding density matrix in the position representation, and let $\rho(x,x',0)$ be the initial density matrix. Using Eqs. (3.15) and (2.7) of Ref. 1(b) one may write

$$\rho(x,x',t_n+) = \int dx_0 \int dx_0' \, \mathcal{J}(x,x' \mid x_0,x_0')\rho(x_0,x_0',0) , \tag{4.14}$$

where the kernel $\mathcal{J}$ is defined by

$$\mathcal{J}(x,x' \mid x_0,x_0') \equiv \int \left[\prod_{r=1}^{n} d\bar{x}_r \right] \mathcal{K}(\{\bar{x}_r\},x \mid x_0)\mathcal{K}^*(\{\bar{x}_r\},x' \mid x_0') . \tag{4.15}$$

Here the "modified propagator" $\mathcal{K}(\{\bar{x}_r\},x \mid x_0)$ has the path-integral expression [Eq. (2.6) of Ref. 1(b)]

$$\mathcal{K}(\{\bar{x}_r\},x \mid x_0) = \int \mathcal{D}x(t) \left[\prod_{r=1}^{n} \Upsilon(\bar{x}_r - x(t_r)) \right] e^{(i/\hbar)S[x(t)]} , \tag{4.16}$$

where $\int \mathcal{D}x(t)$ denotes a Feynman sum over all paths $x(t)$ on the interval $[0,t_n]$ such that $x(0)=x_0$ and $x(t_n)=x$, $\Upsilon(\bar{x}_r - x(t_r))$ is a displaced wave function for the rth meter [Eq. (3.2)], and $S[x(t)]$ is the system's action functional for the path $x(t)$.

We may now write the kernel (4.15) as a double sum over paths,

$$\mathcal{J}(x,x' \mid x_0,x_0') = \int \mathcal{D}x(t) \int \mathcal{D}x'(t) \exp\left[\frac{i}{\hbar}\{S[x(t)]-S[x'(t)]\}\right] I[x(t),x'(t)] , \tag{4.17}$$

where the "influence functional"[11,12] I is defined by

$$I[x(t),x'(t)] \equiv \prod_{r=1}^{n} \left[\int_{-\infty}^{\infty} d\bar{x}_r \, \Upsilon(\bar{x}_r - x(t_r))\Upsilon^*(\bar{x}_r - x'(t_r)) \right] . \tag{4.18}$$

If we now adopt the Gaussian meter wave functions (3.2), we may evaluate the influence functional as

$$I[x(t),x'(t)] = \exp\left[-\frac{1}{4\sigma}\sum_{r=1}^{n}[x(t_r)-x'(t_r)]^2\right] . \tag{4.19}$$

Taking the continuous limit ($n\to\infty$, $\tau\to 0$, with $n\tau=t_n=t$ and $\sigma\tau=D$) we find

$$I[x(t),x'(t)] = \exp\left[-\frac{1}{4D}\int_0^{t_n=t} dt'\,[x(t')-x'(t')]^2\right] . \quad (4.20)$$

Equations (4.14), (4.17), and (4.20) constitute the path-integral representation of the nonselective evolution in the continuous limit; this representation has been given previously by Barchielli, Lanz, and Prosperi.[3]

Equation (4.20) is the influence functional for a Wiener process in momentum with no systematic dissipation.[12] The path-integral picture is thus consistent with the results of the Schrödinger and Heisenberg pictures. The influence functional (4.20) provides perhaps the clearest display of the decay of quantum coherence. Two paths $x(t)$ and $x'(t)$ have a mean-square separation defined by

$$\frac{1}{t}\int_0^t dt'\,[x(t')-x'(t')]^2 ;$$

when this mean-square separation becomes much larger than $4D/t$, then the influence functional (4.20) destroys interference between the two paths in the quantum-mechanical sum (4.17).

These three pictures all show that the nonselective evolution of the system during a continuous measurement of position is given by a Gaussian quantum-dynamical semigroup corresponding to a Wiener process in momentum with no dissipation. In Sec. V we consider how to incorporate dissipation into the measurement model via a feedback mechanism.

V. POSITION MEASUREMENTS WITH FEEDBACK

Our objective now is to modify the basic model developed in preceding sections so as to include systematic dissipation. To motivate the proposed modification reconsider the selective evolution of a free particle analyzed in Sec. III B, and in particular consider the behavior of the mean position and mean momentum. Equations (3.23) show that at each measurement the values of the mean position and mean momentum "jump" in a way that depends on the unpredictable result of the measurement. Thus after many measurements the mean position and mean momentum generally wander a long way from their initial values. In a real measurement process, of course, such a situation cannot arise; the measured system must at least remain in the laboratory with probability close to one. We incorporate this fact explicitly into the description of the measurement process by introducing feedback forces that cancel the jumps in mean position and mean momentum. Formally we modify the operation that specifies the measuring instrument.

Focus attention now on the rth measurement in the sequence. To cancel the jumps in the mean position and mean momentum we introduce feedback forces that act immediately after the rth measurement. Equations (3.30) show that these forces must displace the position by an amount

$$-\frac{C-1}{C}\bar{x}_r \to -\frac{\tau}{t_c}\bar{x}_r \quad (5.1\text{a})$$

and must displace the momentum by an amount

$$-\frac{\hbar}{C^{1/2}\sigma}\bar{x}_r \to -\frac{\hbar\tau}{D}\bar{x}_r . \quad (5.1\text{b})$$

The forms to the right of the arrows apply in the continuous limit; notice that the displacements in position and momentum are proportional to the time τ between measurements.

The displacements of the position and momentum are described formally by a unitary "displacement operator"

$$\hat{D}(\bar{x}_r) \equiv \exp\left[-\frac{i}{\hbar}\tau\bar{x}_r(\gamma_1\hat{x}-\gamma_2\hat{p})\right] = e^{-(i/\hbar)\tau\bar{x}_r\hat{Y}} , \quad (5.2)$$

which is just the evolution operator corresponding to the feedback forces. The operator

$$\hat{Y} \equiv \gamma_1\hat{x}-\gamma_2\hat{p} \quad (5.3)$$

provides a convenient abbreviation. The coefficients γ_1 and γ_2, which specify the size of the displacements, may be parametrized by

$$\gamma_1 \equiv \mu\hbar/D = \mu m/2t_c^2 , \quad (5.4\text{a})$$

$$\gamma_2 \equiv \nu/t_c . \quad (5.4\text{b})$$

We assume throughout the following that $\mu,\nu>0$ (positive feedback). The preferred displacements (5.1) correspond to the choice $\mu=\nu=1$, but for the moment there is no reason to specialize to this preferred case. The effect of the feedback forces is to modify the operation density (3.9) for the rth measurement so that it becomes

$$\begin{aligned}\phi'_{\bar{x}_r}\hat{\rho} &\equiv \hat{D}(\bar{x}_r)\phi_{\bar{x}_r}\hat{\rho}\hat{D}^\dagger(\bar{x}_r)\\ &= \hat{D}(\bar{x}_r)\hat{\Upsilon}(\bar{x}_r)\hat{\rho}\hat{\Upsilon}^\dagger(\bar{x}_r)\hat{D}^\dagger(\bar{x}_r) .\end{aligned} \quad (5.5)$$

To determine the effect of the feedback mechanism we now proceed as in Secs. III A and IV A to derive a master equation for the *nonselective* evolution of the system in the continuous limit, but with the operation density $\phi'_{\bar{x}}$ describing each measurement. Defining the time derivative of $\hat{\rho}(t)$ by Eq. (4.1) we find that Eq. (4.2) is replaced by

$$\begin{aligned}\frac{d\hat{\rho}(t)}{dt} = \lim_{\tau\to 0}\Bigg[&\frac{1}{\tau}\int_{-\infty}^{\infty} d\bar{x}\,\hat{D}(\bar{x})\hat{\Upsilon}(\bar{x})\hat{U}(\tau)\hat{\rho}(t_{n-1}+)\\ &\times\hat{U}^\dagger(\tau)\hat{\Upsilon}^\dagger(\bar{x})\hat{D}^\dagger(\bar{x})\\ &-\frac{1}{\tau}\hat{\rho}(t_{n-1}+)\Bigg] .\end{aligned} \quad (5.6)$$

For short times we may approximate $\hat{D}(\bar{x})$ as

$$\hat{D}(\bar{x}) \simeq \hat{1} - \frac{i}{\hbar}\tau\bar{x}\hat{Y} - \frac{1}{2\hbar^2}\tau^2\bar{x}^2\hat{Y}^2 . \quad (5.7)$$

Proceeding as in Sec. IV A we find that

$$\frac{d\hat{\rho}(t)}{dt} = \left[\frac{d\hat{\rho}(t)}{dt}\right]_{\text{no feedback}} + \lim_{\tau\to 0}\left[-\frac{i}{\hbar}[\hat{Y},\hat{\mathcal{O}}] - \frac{\tau}{2\hbar^2}[\hat{Y},[\hat{Y},\hat{\mathcal{P}}]]\right], \quad (5.8)$$

where the first term is the time derivative in the absence of feedback [Eq. (4.5)], and where the operators $\hat{\mathcal{O}}$ and $\hat{\mathcal{P}}$ are defined by

$$\hat{\mathcal{O}} \equiv \int_{-\infty}^{\infty} d\bar{x}\, \bar{x}\hat{\Upsilon}(\bar{x})\hat{\rho}(t)\hat{\Upsilon}^\dagger(\bar{x}) = \tfrac{1}{2}(\hat{x}\hat{\rho}+\hat{\rho}\hat{x}) + O\left[\frac{1}{\sigma}\right], \quad (5.9a)$$

$$\hat{\mathcal{P}} \equiv \int_{-\infty}^{\infty} d\bar{x}\, \bar{x}^2\hat{\Upsilon}(\bar{x})\hat{\rho}(t)\hat{\Upsilon}^\dagger(\bar{x}) = \tfrac{1}{2}\sigma\hat{\rho} + O(1) \quad (5.9b)$$

[cf. Eq. (4.3)].

Using Eq. (4.5) and the expressions on the right in Eqs. (5.9) (obtained using the results of Gaussian integrals over $\bar{x}^n$) we may put the master equation in its final form,

$$\frac{d\hat{\rho}(t)}{dt} = -\frac{i}{\hbar}[\hat{H}_0,\hat{\rho}(t)] - \frac{i}{2\hbar}[\hat{Y},\hat{x}\hat{\rho}(t)+\hat{\rho}(t)\hat{x}] - \frac{1}{4D}[\hat{x},[\hat{x},\hat{\rho}(t)]] - \frac{D}{4\hbar^2}[\hat{Y},[\hat{Y},\hat{\rho}(t)]] . \quad (5.10)$$

The feedback gives rise to two new terms in the master equation, the second and the fourth. The second term describes two systematic effects, a linear restoring force and systematic dissipation. The fourth term describes additional fluctuations that arise as a consequence of the dissipation.

Insight into the master equation (5.10) comes from considering the example of a free particle. The equations of motion for the mean position and mean momentum are

$$\frac{d\langle\hat{x}\rangle}{dt} = \frac{\langle\hat{p}\rangle}{m} - \gamma_2\langle\hat{x}\rangle , \quad (5.11a)$$

$$\frac{d\langle\hat{p}\rangle}{dt} = -\gamma_1\langle\hat{x}\rangle . \quad (5.11b)$$

In addition to introducing position damping (coefficient γ_2), the feedback transforms the dynamics (coefficient γ_1) so that the origin in phase space becomes an elliptic fixed point. The dynamics of the first-order moments becomes that of a damped harmonic oscillator with resonant frequency $(\gamma_1/m)^{1/2} = (\mu/2)^{1/2}t_c^{-1}$ and amplitude damping constant $\gamma_2/2 = (\nu/2)t_c^{-1}$.

The second-order moments obey the following equations of motion:

$$\frac{d\langle\hat{x}^2\rangle}{dt} = \frac{1}{m}\langle\hat{x}\hat{p}+\hat{p}\hat{x}\rangle - 2\gamma_2\langle\hat{x}^2\rangle + \tfrac{1}{2}D\gamma_2^2 , \quad (5.12a)$$

$$\frac{d\langle\hat{p}^2\rangle}{dt} = -\gamma_1\langle\hat{x}\hat{p}+\hat{p}\hat{x}\rangle + \frac{\hbar^2}{2D} + \tfrac{1}{2}D\gamma_1^2 , \quad (5.12b)$$

$$\frac{d\langle\hat{x}\hat{p}+\hat{p}\hat{x}\rangle}{dt} = \frac{2}{m}\langle\hat{p}^2\rangle - 2\gamma_1\langle\hat{x}^2\rangle - \gamma_2\langle\hat{x}\hat{p}+\hat{p}\hat{x}\rangle + D\gamma_1\gamma_2 . \quad (5.12c)$$

The first- and second-moment equations have a stationary solution. The mean position and mean momentum damp to zero, and the second moments are written most compactly in terms of the dimensionless parameters μ and ν of Eqs. (5.4),

$$\langle\hat{x}^2\rangle = [\tfrac{1}{2}\nu + \tfrac{1}{4}\nu^{-1}(\mu+\mu^{-1})]\tfrac{1}{2}\Delta_c , \quad (5.13a)$$

$$\langle\hat{p}^2\rangle = [\tfrac{1}{2}\nu\mu^{-1} + \tfrac{1}{4}\mu\nu^{-1}(\mu+\mu^{-1})]\frac{\hbar^2}{\Delta_c} , \quad (5.13b)$$

$$\langle\hat{x}\hat{p}+\hat{p}\hat{x}\rangle = [\tfrac{1}{2}(\mu+\mu^{-1})]\hbar . \quad (5.13c)$$

For the preferred feedback strength $\mu=\nu=1$ the factors in square brackets in Eqs. (5.13) are equal to unity. Thus the covariance matrix (5.13) is identical to that for stationary selective evolution in the continuous limit [Eqs. (3.17) and (3.31); $\Delta=\Delta_c$ and $\epsilon=1$]. The feedback suppresses completely the jumps in the mean position and mean momentum which occur in the selective evolution; thus the covariance matrix for nonselective evolution reduces to that for selective evolution. An equivalent statement is that the state of the free particle damps to the Gaussian pure state (3.15) with zero mean position and zero mean momentum ($a=b=0$) and with $\Delta=\Delta_c$ and $\epsilon=1$. There is no decay of off-diagonal coherence in the position basis. One might guess, however, that there is a decay of quantum coherence in an overcomplete coherent-state-like basis consisting of the Gaussian pure states (3.15) with $\Delta=\Delta_c$ and $\epsilon=1$; this decay would be like the decay of quantum coherence in the coherent state basis for a linearly damped harmonic oscillator.[22]

VI. CONCLUSION

In this paper we have constructed a model for continuous measurements of position. The model is constructed by considering a sequence of position measurements of accuracy $(\sigma/2)^{1/2}$ separated by a time τ. The constant $D=\sigma\tau$ plays a crucial role throughout the analysis. Making D smaller means making more accurate measurements more often.

The position measurements disturb the momentum of the system, which diffuses as $(\hbar^2/2D)^{1/2}t^{1/2}$. Making D smaller means a greater momentum disturbance more often. To take the continuous limit one must let $\tau\to 0$ while simultaneously letting $\sigma\to\infty$ in such a way that $D=$const. This limiting procedure is chosen so that the momentum disturbance during a fixed time interval is held constant.

We have analyzed in detail both the selective and nonselective evolution of a free particle subjected to continuous measurements. Selective evolution corresponds to the case where the results of the measurements are recorded. We find that the selective evolution approaches a stationary configuration in which the position variance takes on the value $\frac{1}{2}\Delta_c=(\hbar/2m)^{1/2}D^{1/2}$. The smaller the value of D, the more precisely the measurements localize the position of the particle.

Nonselective evolution corresponds to the case where the results of the measurements are discarded. To determine the nonselective evolution one averages over the possible results. The analysis shows that the measurements lead to a decay of quantum coherence in the position basis. Quantum interference between positions a distance L apart is destroyed on a time scale $4D/L^2$. The smaller the value of D, the more rapidly quantum coherence is destroyed. In the language of Zurek[5] the measurements establish position as the pointer basis.

Finally we have modified the basic model by including a feedback mechanism to suppress the wandering of the position and momentum. The feedback converts a free particle into a damped harmonic oscillator. This oscillator damps to a stationary state that has position variance $\frac{1}{2}\Delta_c=(\hbar/2m)^{1/2}D^{1/2}$, the same variance that applies to the stationary selective evolution in the basic model.

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APPENDIX A: LINEAR STABILITY OF STATIONARY SELECTIVE EVOLUTION

In this appendix we demonstrate that the stationary solution (3.27) is stable under linear perturbations. We start with the general equations for the evolution of the second-moment parameters,

$$C_r\Delta_{r+1}=\Delta_r+\frac{1+\epsilon_r^2}{\Delta_r}\left[\frac{\hbar\tau}{m}\right]^2+2\epsilon_r\frac{\hbar\tau}{m}\ , \tag{A1a}$$

$$C_r\epsilon_{r+1}=\epsilon_r+\frac{1+\epsilon_r^2}{\Delta_r}\frac{\hbar\tau}{m}\ , \tag{A1b}$$

$$C_r=(1-\Delta_{r+1}/\sigma)^{-1} \tag{A1c}$$

[Eqs. (3.19), (3.24), and (3.25)], and then perturb about the stationary configuration by writing

$$\Delta_r=\Delta(1+x_r)\ , \tag{A2a}$$

$$\epsilon_r=\epsilon(1+y_r)\ . \tag{A2b}$$

Linearizing Eqs. (A1) in x_r and y_r, we find that

$$\begin{bmatrix} x_{r+1} \\ y_{r+1} \end{bmatrix}=\underline{M}\begin{bmatrix} x_r \\ y_r \end{bmatrix}\ , \tag{A3a}$$

$$\underline{M}\equiv-\frac{1}{C(C+1)^2}\begin{bmatrix}(C-3)(C+1) & -4(C-1) \\ 4(C-1)(C+1) & C^2-10C+5\end{bmatrix}\ . \tag{A3b}$$

The eigenvalues of $\underline{M}$, given by

$$\lambda_\pm=-\frac{1}{C(C+1)^2}[C^2-6C+1\pm 4iC^{1/2}(C-1)]\ , \tag{A4}$$

have magnitude

$$|\lambda_\pm|=1/C\ . \tag{A5}$$

Since the conditions for linear stability are $|\lambda_\pm|<1$, the stationary configuration is stable if $C>1$.

One can also use this analysis to investigate the approach to stationarity in the continuous limit. After n measurements lasting a time $t=n\tau$, the decay of the perturbation is governed by the factor $|\lambda_\pm|^n=C^{-n}$. In the continuous limit this factor becomes an exponential decay,

$$|\lambda_\pm|^n=\left[1+\frac{C-1}{\tau}\frac{t}{n}\right]^{-n}\to e^{-t/t_c}\ , \tag{A6}$$

where the e-folding time t_c is defined in Eq. (3.31a).

APPENDIX B: DECAY OF OFF-DIAGONAL COHERENCE

In this appendix we consider the solution of Eq. (4.9) for a free particle and show how it leads to rapid decay in the off-diagonal elements of $\rho(x,x',t)$. The free-particle Hamiltonian is $\hat{H}_0=\hat{p}^2/2m$, so Eq. (4.9) becomes

$$\frac{\partial\rho(x,x',t)}{\partial t}=i\mu\left[\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial x'^2}\right]\rho(x,x',t) -\gamma(x-x')^2\rho(x,x',t)\ , \tag{B1}$$

where $\mu\equiv\hbar/2m$ and $\gamma\equiv 1/4D$. We now make the change of variables

$$u\equiv\tfrac{1}{2}(x+x')\ ,\quad v\equiv x-x'\ .$$

In these variables Eq. (B1) becomes

$$\frac{\partial\rho(u,v,t)}{\partial t}=2i\mu\frac{\partial^2\rho(u,v,t)}{\partial u\,\partial v}-\gamma v^2\rho(u,v,t)\ . \tag{B2}$$

Taking the Fourier transform with respect to u,

$$\tilde{\rho}(\alpha,v,t)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}du\ e^{-i\alpha u}\rho(u,v,t)\ ,$$

Eq. (B2) becomes

$$\frac{\partial\tilde{\rho}(\alpha,v,t)}{\partial t}+2\mu\alpha\frac{\partial\tilde{\rho}(\alpha,v,t)}{\partial v}=-\gamma v^2\tilde{\rho}(\alpha,v,t)\ . \tag{B3}$$

Equation (B3) may be solved by the method of characteristics to give

$$\tilde{\rho}(\alpha,v,t)=\tilde{\rho}_0(\alpha,v-2\mu\alpha t)\times\exp[-\gamma t(v^2-2\mu\alpha v t+\tfrac{4}{3}\mu^2\alpha^2t^2)]\ , \tag{B4}$$

where $\tilde{\rho}_0(\alpha,v)\equiv\tilde{\rho}(\alpha,v,0)$ is the initial condition.

We now specialize to a Gaussian initial state with wave function

$$\psi_0(x)=(\pi\Delta_0)^{-1/4}\exp(-x^2/2\Delta_0) .$$

In this case the solution becomes

$$\tilde{\rho}(u,v,t)=[\pi A(t)]^{-1/2}\exp\left[-\frac{u^2}{A(t)}-R(t)v^2+\frac{P(t)}{A(t)}2iuv\right] , \tag{B5}$$

where

$$A(t)\equiv\Delta_0+4\mu^2t^2\left[\frac{1}{\Delta_0}+\frac{4}{3}\gamma t\right] , \tag{B6a}$$

$$P(t)\equiv\mu t\left[\frac{1}{\Delta_0}+2\gamma t\right] , \tag{B6b}$$

$$R(t)\equiv\frac{1}{4\Delta_0}+\gamma t-\frac{P^2(t)}{A(t)} . \tag{B6c}$$

The function $R(t)$ governs the decay of off-diagonal coherence.

The analysis in Sec. III B makes clear the crucial role played by the stationary value

$$\Delta_c\equiv(2\hbar/m)^{1/2}D^{1/2}=(\mu/\gamma)^{1/2} \tag{B7}$$

[Eq. (3.31b)], so we write

$$\Delta_0=\delta\Delta_c=\delta(\mu/\gamma)^{1/2} , \tag{B8}$$

where δ is dimensionless. Furthermore, the analysis in Appendix A reveals a characteristic time

$$t_c\equiv(m/2\hbar)^{1/2}D^{1/2}=\tfrac{1}{4}(\mu\gamma)^{-1/2} \tag{B9}$$

[Eq. (3.31a)], at which, for $\delta\simeq 1$, the three terms in $A(t)$ are of the same size and the two terms in $P(t)$ are of the same size. Hence we define a dimensionless time

$$\eta\equiv t/t_c . \tag{B10}$$

In terms of these dimensionless quantities Eqs. (B6) become

$$A(t)=\left[\frac{\mu}{\gamma}\right]^{1/2}[\delta+\tfrac{1}{4}\eta^2(\delta^{-1}+\tfrac{1}{3}\eta)] , \tag{B11a}$$

$$P(t)=\tfrac{1}{4}\eta(\delta^{-1}+\tfrac{1}{2}\eta) , \tag{B11b}$$

$$R(t)=\frac{1}{4}\left[\frac{\gamma}{\mu}\right]^{1/2}(\delta^{-1}+\eta)-\frac{P^2(t)}{A(t)} . \tag{B11c}$$

In the limit of short times, $\eta\ll\delta^{-1}$, one may say quite generally that the evolution is that of an unmeasured free particle ($\gamma=0$). In the opposite limit, $\eta\gg\delta^{-1}$, one must consider two cases, distinguished by whether δ is bigger or smaller than unity.

Case (i). $\delta\lesssim 1$. The particle is initially well localized on the scale set by Δ_c. There is a single long time limit, $\eta\gg\delta^{-1}\gtrsim 1\gtrsim\delta^{1/3}$; one easily shows that in this limit $R(t)=\frac{1}{4}\gamma t$. Thus the damping constant for off-diagonal coherence has one-fourth the naive value inferred in Sec. IV A.

Case (ii). $\delta\gtrsim 1$. The particle is not initially well localized on the scale set by Δ_c. There is a region of intermediate times, $\delta^{-1}\ll\eta\ll\delta^{1/3}$, if $\delta\gg 1$; for such intermediate times one finds that $R(t)=\gamma t$. There is in addition a long time limit, $\eta\gg\delta^{1/3}\gtrsim\delta^{-1}$, for which $R(t)=\frac{1}{4}\gamma t$. Thus in this case off-diagonal coherence decays initially with the expected decay constant before switching to the slower decay found in case (i). The particle not being initially well localized in position, the model apparently "works harder" initially to kill off the quantum coherence.

APPENDIX C: TWO-TIME CORRELATION FUNCTION

In this appendix we derive in the continuous limit the two-time correlation function for the Langevin force

$$\hat{F}_p(t')=-\sum_{r=1}^{n}\delta(t'-r\tau)\hat{\bar{p}}_r \tag{C1}$$

[Eq. (4.11)]. Recall that $t=n\tau$ is the duration of the sequence of measurements; t' in Eq. (C1) is a time during the sequence. The two-time correlation function is defined by

$$G_p(t',t'')\equiv\langle\hat{F}_p(t')\hat{F}_p(t'')\rangle . \tag{C2}$$

Representing $\delta(t-r\tau)$ as a Fourier integral, we may write

$$G_p(t',t'')=\int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}\sum_{r,s=1}^{n}e^{-i\omega_1(t'-r\tau)}e^{i\omega_2(t''-s\tau)}\langle\hat{\bar{p}}_r\hat{\bar{p}}_s\rangle . \tag{C3}$$

The meter wave functions (3.2) imply that

$$\langle\hat{\bar{p}}_r\hat{\bar{p}}_s\rangle=\frac{\hbar^2}{2\sigma}\delta_{rs} , \tag{C4}$$

which leads to

$$G_p(t',t'')=\frac{\hbar^2}{2}\int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}e^{-i\omega_1t'}e^{i\omega_2t''}\left[\frac{1}{\sigma}\sum_{r=1}^{n}e^{i(\omega_1-\omega_2)r\tau}\right] . \tag{C5}$$

The sum may be evaluated directly. If we then take the continuous limit ($n \to \infty$, $\tau \to 0$, with $n\tau = t$ and $\sigma\tau = D$), the factor in large parentheses in Eq. (C5) becomes

$$\frac{1}{D}\int_0^t dv\, e^{i(\omega_1-\omega_2)v} .$$

Thus we obtain

$$G_p(t',t'')=\frac{\hbar^2}{2D}\int_0^t dv\, \delta(t'-v)\delta(t''-v)=\frac{\hbar^2}{2D}\delta(t'-t'') . \tag{C6}$$

[1](a) C. M. Caves, Phys. Rev. D **33**, 1643 (1986); (b) **35**, 1815 (1987).

[2]C. M. Caves, in *Quantum Measurement and Chaos,* edited by E. R. Pike and S. Sarkar (Plenum, New York, 1987).

[3]A. Barchielli, L. Lanz, and G. M. Prosperi, Nuovo Cimento B **72**, 79 (1982); Found. Phys. **13**, 779 (1983).

[4]A. Barchielli, Nuovo Cimento B **74**, 113 (1983).

[5]W. H. Zurek, Phys. Rev. D **24**, 1516 (1981); **26**, 1862 (1982).

[6]E. Joos and H. D. Zeh, Z. Phys. B **59**, 223 (1985).

[7]G. Ludwig, *Foundations of Quantum Mechanics I and II* (Springer, Berlin, 1983).

[8]K. Kraus, *States, Effects, and Operations: Fundamental Notions of Quantum Theory* (Springer, Berlin, 1983).

[9]E. Prugovečki, *Stochastic Quantum Mechanics and Quantum Spacetime* (Reidel, Dordrecht, 1984).

[10]H. Spohn, Rev. Mod. Phys. **52**, 569 (1980).

[11]R. P. Feynman and F. L. Vernon, Jr., Ann. Phys. (N.Y.) **24**, 118 (1963).

[12]A. O. Caldeira and A. J. Leggett, Physica A **121**, 587 (1983).

[13]J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932), especially Chap. 6 [English translation: *Mathematical Foundations of Quantum Mechanics* (Princeton University, Princeton, New Jersey, 1955)].

[14]Y. Aharonov and D. Bohm, Phys. Rev. **122**, 1649 (1961).

[15]B. d'Espagnat, Found. Phys. **16**, 351 (1986).

[16]S. L. Braunstein and C. M. Caves, Found. Phys. Lett. (to be published).

[17]B. L. Schumaker, Phys. Rep. **135**, 318 (1986).

[18]W. E. Lamb, Jr., in *Quantum Measurement and Chaos,* edited by E. R. Pike and S. Sarkar (Plenum, New York, 1987).

[19]H. Dehmelt, Proc. Natl. Acad. Sci. USA **83**, 2291 (1986); **83**, 3074 (1986).

[20]G. C. Ghirardi, A. Rimini, and T. Weber, Phys. Rev. D **34**, 470 (1986).

[21]A. O. Caldeira and A. J. Leggett, Phys. Rev. A **31**, 1059 (1985).

[22]D. F. Walls and G. J. Milburn, Phys. Rev. A **31**, 2403 (1985).

[23]C. M. Savage and D. F. Walls, Phys. Rev. A **32**, 2316 (1985); **32**, 3487 (1985).

[24]F. Haake and D. F. Walls, Phys. Rev. A **36**, 730 (1987); see also F. Haake and D. F. Walls, in *Quantum Optics IV,* edited by J. D. Harvey and D. F. Walls (Springer, Berlin, 1986), p. 181.

[25]H. Haken, *Light,* (North-Holland, Amsterdam, 1981), Vol. I, p. 297.