Fault Tolerant Quantum Computation with Nondeterministic Gates

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In certain approaches to quantum computing the operations between qubits are nondeterministic and likely to fail. For example, a distributed quantum processor would achieve scalability by networking together many small components; operations between components should be assumed to be failure prone. In the ultimate limit of this architecture each component contains only one qubit. Here we derive thresholds for fault-tolerant quantum computation under this extreme paradigm. We find that computation is supported for remarkably high failure rates (exceeding 90%) providing that failures are heralded; meanwhile the rate of unknown errors should not exceed 2 in $10^4$.

Quantum information processing (QIP) has seen many experimental successes, however scaling from a few qubits to many remains unsolved. This issue is so crucial that it may dictate the fundamental architecture for the machine. For example, in the concept of distributed QIP a plurality of small components are networked together to constitute a full scale machine. The components may be trapped atoms or ions, or solid state nanostructures such as quantum dots or NV centers. Each component can be presumed to be under good control, and it is understood that the key task is then to entangle the physically remote components. One method of achieving this entangling operation (EO) is to arrange for each component to emit a photon correlated with the internal state of the component, before performing a joint measurement of the photons. A considerable number of such entangling schemes have been advanced since the first ideas in 1999 [3,4]. Notably, photon loss can be detected, or heralded, within such a protocol [5,6]. In these remote entangling protocols, one is supposed to employ optical measurements that simultaneously observe two, or four, components. This principle for generating entanglement has been demonstrated with ensemble systems [8] and with individual atoms [9].

Remote EOs may be failure prone. However, these failures are assumed to be heralded: the experimentalist is aware when a failure occurs. In the case that each component incorporates multiple qubits then we can nominate one “logical qubit” and use the other(s) to make repeated attempts at remote entanglement; when we are eventually successful then we transfer the entanglement to the logical qubits [10,11]. However, many physical systems may have very limited complexity, and moreover it is always desirable to minimize the required complexity. Therefore, we consider the case of just one qubit in each component. This is the extreme limit of the distributed paradigm. Suppose that the probability $p_\text{h}$ of a heralded error is high, perhaps well above 50%, then we cannot perform quantum computation by directly implementing a standard circuit model approach. It has been shown that in spite of such heralded failures, arbitrary quantum algorithms can be implemented [5,6,12–16]. These insights are related to earlier ideas on photonic QIP [17,18]. While such schemes demonstrated that large heralded failure rates can be tolerated, this was not shown in a fully fault-tolerant manner. In particular, it was not known if large heralded failure rates can be tolerated in the presence of realistic error rates for all other elementary operations.

Other studies have developed an approach which can be adapted to present purposes. A series of beautiful results by Raussendorf, Harrington and others described a method for QIP by creating a cubic lattice cluster state [19,19,20]. Defect regions within the 3D lattice are braided together, yielding topologically protected Clifford gates. QIP implemented using this topologically protected cluster (TPC) state has a remarkably large tolerance against elementary errors (at rates $\leq 1\%$) during preparation, entangling operations and single-qubit measurement. Subsequently, we extended this idea to incorporate the possibility that the lattice contains a significant proportion of heralded missing qubits (nearly 25% can be lost) [21–23].

Here we consider the generation of a TPC state when the entangling operations are themselves subject to heralded failures during the cluster state growth process. The result is a lattice with a certain proportion of known failed entanglement relations (missing “edges” in the graph state). Determining a threshold for universal QIP depends on the proper choice of growth strategy and a careful audit of the accumulation of unknown errors in that process. We map this cluster state with missing “edges” to one with missing qubits, and use the loss-tolerant thresholds quoted [21–23].
Several papers have previously considered the task of creating large entangled states using a failure prone EO (see Fig. 1 and caption). A “divide and conquer” approach enlarges the cluster state on average for any nonzero success probability \( p_s = 1 - p_h \) [5,6,12–16]. Generally the solution involves generating small resource states and subsequently connecting them. As shown in Fig. 1(a) the possible resources include stars [13], linear clusters [5,12] which give rise to cross structures [14], and tree topologies [24]. The “snowflake” is an optimal choice for minimizing errors [16].

In this Letter we synthesize the TPC state [Fig. 1(b), inset top left]. This structure has a connectivity of 4. Therefore, we attempt to entangle each resource with four others, Fig. 1(b). In this example there will be \( N = 4 \) attempts to connect to each of the surrounding snowflakes. If one or more of these attempts succeeds, the snowflakes are successfully connected, while with probability \( p_h^N \) all attempts will fail and the resulting TPC state will have a missing “edge”. These missing edges are known, and therefore are imperfections which we account for subsequently. It will be necessary to create resource objects which are sufficiently large that this failure probability is below the threshold for fault-tolerant QIP, discussed below. For high values of \( p_h \) we will see that the resource states must be considerably larger than those in Fig. 1.

To evaluate this scheme, we determine the accumulation of unknown errors when we perform the star, cross, and snowflake strategies. These errors occur during growth, fusion [as in Fig. 1(b)], and when removing redundant qubits to simplify down to the TPC state. To minimize error accumulation during growth we abandon the entire object when a failure is detected. Fortunately, all three of the resources we consider can be grown through a series of steps each of which (on success) doubles the entity’s size. Thus the process is quick in the sense that the number of successful steps is merely a logarithmic function of the target resource size. Here we assume each resource is grown using one of two forms of EO: we use either parity projections, i.e., projecting a pair of qubits into the odd or even parity subspaces, or a canonical control-phase gate between the qubit pair, depending on which is more efficient [25]. Both operations are known to be possible through suitable measurements on emitted photons [5,6,12] and atomic ensembles [26].

Single-qubit errors may occur during preparation, while performing a single-qubit rotation, or during measurement. These errors may also occur passively in memory. Meanwhile two-qubit errors may occur during entangling operations. We account for imperfections both in the emission of photons and errors arising from imperfect measurement of emitted photons [25]. For simplicity of discussion, we set the rates for all forms of the active “gate” errors to be equal and we denote their probability \( p_G \). Memory errors are considered later.

A conclusion of our analysis of error propagation is that two-qubit errors occurring during the growth and fusion of resource objects typically appear as single-qubit errors in the eventual TPC state. While a two-qubit error can afflict the ultimate lattice, the majority of these involve one qubit from the prime lattice and one from the dual. Such correlations do not affect the fault tolerance threshold. There will be occasional instances of errors between two qubits both within the prime lattice, or both in the dual. However these are rare—for example, in the case where one uses the snowflake strategy with \( p_h = 0.9 \), the rate for these errors is 2 orders of magnitude lower than the corresponding rate of single-qubit errors on the TPC (given equal rates for the various forms of error during growth) [25]. In this case,
we ignore such events and assume that all gate errors affect at most one qubit in each sublattice. Thus we consider a lattice with a (low) rate of random single-qubit errors, and a (relatively high) portion of missing “edges.” We determine the threshold for such a lattice to support computation. Our previous work has considered the closely related case of a lattice with many missing nodes. We thus map the case of missing edges to that of missing nodes and obtain thresholds in the present case.

Consider the standard TPC state, specifically neighboring qubits $i$ (in the primal lattice) and $j$ (in the dual lattice) distinguished in Fig. 1(b) by open circles versus filled circles. Each qubit is centered on a face of its respective sublattice and is a member of two cubic unit cells of the sublattice. Ideally where no bonds are missing, the product of cluster stabilizers associated with the faces of each cubic unit cell is simply the product of $X$ operators acting on the respective face-centered qubits, yielding two parity-check operators associated with each qubit: $\hat{P}_{i}^{1,2}$ for qubit $i$ and $\hat{P}_{j}^{1,2}$ for qubit $j$. Since these ideal parity-check operators are just products of $X$ operators on each face of the corresponding cube, they commute pointwise, which enables the error syndrome to be determined by single particle $X$ measurements [19,19,20].

When the bond between qubits $i$ and $j$ is missing, cluster stabilizers associated with the missing bond are modified. Then, the product of cluster stabilizers centered on the cubic unit cell faces yields damaged parity-check operators $\hat{P}_{i}^{1,2} = \hat{P}_{i}^{1,2} Z_{j}$ and $\hat{P}_{j}^{1,2} = \hat{P}_{j}^{1,2} Z_{i}$. While $\hat{P}_{i}^{1,2}$ and $\hat{P}_{j}^{1,2}$ commute, they do not commute pointwise (since $[X_{i}, Z_{j}] \neq 0$). In contrast to the ideal case, this means that determining the syndrome on the primal and dual lattices apparently requires measurement of the two-qubit operators $X_{i} Z_{j}$ and $Z_{i} X_{j}$.

Fortunately, by simply treating the qubits $i$ and $j$ at each end of the missing bond as lost, and adopting the strategy in [21–23], we form products of the damaged parity-check operators, yielding supercheck operators $\hat{P}_{i} = \hat{P}_{i}^{1,2} \hat{P}_{j}^{1,2} = \hat{P}_{i}^{1} \hat{P}_{j}^{2}$ and $\hat{P}_{j} = \hat{P}_{i}^{1,2} \hat{P}_{j}^{1,2} = \hat{P}_{i}^{2} \hat{P}_{j}^{1}$. These new operators are independent of the qubits $i$ and $j$, so are unaffected by the missing bond between them. Furthermore each supercheck operator involves only products of $X$ operators from a single sublattice, so a missing bond manifests itself as a single missing qubit on each sublattice. This establishes a correspondence between missing bonds and correlated losses of neighboring qubits. Error correction is then realized by implementing the loss-tolerant, error-correcting protocol of [21–23] to each sublattice independently [27].

Having made the connection to prior work on thresholds for the TPC state, we can now take parameters for the low-level operations on qubits in the distributed machine, compute the effective qubit loss rate, and determine whether quantum computation is possible. Figure 2 shows this phase diagram assuming that all gate error rates are equal. Very high rates of heralded error can be tolerated, provided

that the rate for unknown errors is below $2 \times 10^{-4}$. This is a low target but might be possible in some implementations, e.g., trapped ions for which multiqubit measurements with fidelity around this rate (in a single basis) have already been demonstrated [29].

We assume memory errors happen at a lower rate than gate errors. Figure 3(a) shows the effect of “switching on” memory errors at 10% of the gate error rate. This lowers the overall threshold, but not dramatically.

Finally, we consider the question of resource scaling. From Fig. 2 one might be tempted to conclude that QIP is possible with extremely high rates of heralded error, above 99% or more. However, such a conclusion would neglect the ever increasing costs of preparing the resource objects. These objects become large as $p_{h} \rightarrow 1$. In Fig. 3(b) we see that if $p_{h}$ exceeds 0.98, the size of each snowflake must be several thousand qubits. Recall that each snowflake ultimately corresponds to a single node in the TPC state, and therefore this factor would multiply the overhead already implicit in that approach. However, values in the range of $p_{h} \approx 0.9$ may be tenable for technologies where the individual components of the distributed computer can be mass produced.

In conclusion, we have determined the threshold for quantum computation for nondeterministic two-qubit gates, e.g., a network of components each of which contains only a single qubit. We find that it is acceptable for entangling operations over the network to fail with probability exceeding 90% provided that these failures are her-
altered and the rate of unknown errors does not exceed 0.02%. Our analysis sets a target for experimental single-qubit components.

Note added.—Since the initial submission of this Letter another similar preprint was made public [30]. It tackles the same objectives with a comparable approach; while it employs a different technique for error suppression, the authors’ conclusions are broadly in line with our own.

[27] In principle, the correlated nature of the losses between sublattices could be exploited using belief propagation methods [28].