Correspondence

Note on the Use of the Wigner Distribution for Time-Frequency Signal Analysis

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***Abstract*—We show that a correct use of the Wigner Distribution (WD) for time-frequency signal analysis requires use of the analytic signal. This version, often referred to as the Wigner–Ville Distribution (WVD), is straightforward to compute, does not exhibit any aliasing problem, and introduces no frequency artifacts. The problems introduced by the use of the Wigner Distribution with a real signal are clarified.**

I. Introduction

The Wigner Distribution has been investigated by several authors as a tool for time-frequency signal analysis, and several applications have been proposed in different domains.

Although it has a unique origin in quantum mechanics, in signal processing the different versions proposed differ slightly. It was first defined by Ville in 1948 [1] using the analytic signal, and later applied to signal analysis [2]–[4]. Other authors have also used the Wigner Distribution with the real signal (WD) [5]–[7].

The first objective of this correspondence is to show that, for a better time-frequency signal analysis, one should return to the original definition that uses the analytic signal. This version is often referred to as the Wigner–Ville Distribution (WVD) [8].

Definition 1: The WVD of a real signal $s(t)$ is expressed as follows:

$$W(t,f) = \int_{-\infty}^{+\infty} z\left(t+\frac{\tau}{2}\right) \cdot z^*\left(t-\frac{\tau}{2}\right) \cdot e^{-j2\pi f\tau} \cdot d\tau \quad (1)$$

where $z(t)$ is the analytic signal associated with the real signal $s(t)$. It is calculated in the time domain as follows:

$$z(t) = s(t) + j \cdot H[s(t)]$$

where $H[\cdot]$ stands for the Hilbert Transform; or it can be calculated in the frequency domain using the Fourier transforms $Z(f)$ and $S(f)$ of $z(t)$ and $s(t)$ as follows:

$$\begin{aligned} Z(f) &= 2 \cdot S(f), \quad & f > 0 \\ &= S(f), & f = 0 \\ &= 0, & f < 0. \end{aligned} \quad (2)$$

Definition 2: The Wigner Distribution (WD) of a real signal $s(t)$ is expressed as follows:

$$\rho(t,f) = \int_{-\infty}^{+\infty} s\left(t-\frac{\tau}{2}\right) \cdot s\left(t+\frac{\tau}{2}\right) \cdot e^{-j2\pi f\tau}\, d\tau. \quad (3)$$

The next sections present the problem of computing the WVD and the WD without aliasing.

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II. Artifacts in the Wigner Distribution

Lemma: The Wigner Distribution (WD) of a signal $s(t)$ defined by (3) does not in general represent the true time-frequency pattern of this signal.

Example 1: The chirp signal is expressed as

$$s_1(t) = \Pi_T(t) \cdot \cos 2\pi(f_0 \cdot t + \alpha \cdot t^2/2) \quad \text{with } \alpha = B/T \quad (4)$$

where

$$\begin{aligned} \Pi_T(t) &= 1, \quad \text{for } |t| < T/2 \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

The WVD of such a signal is expressed as [2]

$$W(t,f) = 2(T - 2|t|) \cdot \Pi_T(t) \cdot \text{sinc}\left[(T-2|t|)\,(f - f_i(t))\right] \quad (5)$$

where

$$\text{sinc}\, x = \frac{\sin \pi x}{\pi x}.$$

The WVD then exhibits a concentration of energy along the instantaneous frequency $f_i(t)$ of the signal that represents here the FM law [2]

$$f_i(t) = f_0 + \alpha t. \quad (6)$$

Now, to calculate $\rho(t,f)$, we expand $s_1(t) = \cos[\cdot]$ in complex exponentials. It was shown [2] that the WD of the chirp signal $s_1(t)$ can be best approximated by

$$\rho(t,f) = \frac{1}{4}\left(W(t,f) + W(t,-f)\right) + \frac{1}{2\sqrt{\alpha}} \cdot \Pi_T(t) \cdot \Pi_{B(1-2|t/t|)}(f) \cdot \cos 2\pi\left(\frac{f^2}{\alpha} - 2f_0 \cdot t - \alpha \cdot t^2\right). \quad (7)$$

This result shows that the Wigner Distribution of such a modulated signal is the sum of its WVD plus some interaction components causing artifacts, and therefore does not exhibit a pattern concentrated along the time-frequency law $f_i(t)$ of the signal.

Example 2: If we compare the WVD and the WD on a signal $s_2(t)$ that is FM with an hyperbolic law, we obtain a similar result. A WIGNER APL program [2], [3] allowed the simulations given in Figs. 7–10. These results are in agreement with (7) and suggest that the analytic signal should be used for a better time-frequency analysis. This was also pointed out in [9].

The next sections present the problem of computing the discrete-time versions of the WD and the WVD.

III. Discrete-Time Version of the Wigner Distribution (WD)

If we let $\theta = \tau/2$ in (3), we obtain the form

$$\rho(t,f) = 2 \cdot \int_{-\infty}^{+\infty} s(t+\theta) \cdot s(t-\theta) \cdot e^{-4\pi jf\theta} \cdot d\theta \quad (8)$$

$$\rho(t,f) = \left\{2 \cdot \underset{\theta \to 2f}{FT}\left(s(t+\theta) \cdot s(t-\theta)\right)\right\}. \quad (9)$$

Suppose $s(t)$ is a band-limited signal sampled at the Nyquist rate [twice the maximum frequency of $s(t)$] resulting in a sequence of duration N, $s(n)$.

A direct application of the integral form 3 to the discrete case leads to the following expression:

$$\rho(t, f) = \rho(n \cdot \Delta t, k \cdot \Delta f) \triangleq W(n, k)$$
$$= 2 \cdot \underset{m \to 2k}{\mathrm{DFT}} \left[s(n + m) \cdot s(n - m) \right]$$
$$(\text{DFT} = \text{Discrete Fourier Transform}) \qquad (10)$$

where Δt and Δf are, respectively, the time sampling interval and the frequency sampling interval chosen correctly (as described below).

Suppose now that f max is the frequency of the highest frequency component of the incoming signal. If $s(n)$ is sampled at the Nyquist rate which is $2 \cdot f$ max, then no samples are available at $s(n + m/2)$ as is required for the WD.

This can be rectified in two ways:

1) by requiring that $s(t)$ be oversampled at twice the Nyquist rate [11] (however, it is not in general possible to increase the sampling rate of the "available" digital data, as the analog signal is not available anymore); or

2) by use of interpolation easily obtained by adding N zeros to the result of the DFT of $s(n)$ [12] (we do not lose or add any information by doing so).

Unfortunately, low-frequency artifacts are still present in the Wigner Distribution, as shown in Section II. These low-frequency artifacts have no physical meaning. They have nothing to do with the method of interpolation or the sampling rate, as they are caused by interaction between positive and negative frequencies. We show below that all these problems do not occur when one uses the Wigner–Ville Distribution, i.e., the analytic signal. (Of course, it should be noted that a single sample of the analytic signal $z(t)$ consists of a real part and an imaginary part which, intuitively, should be as effective as two real samples of the real signal $s(t)$.)

IV. Aliasing Problem?

Consider $s(t)$, which when sampled at the Nyquist rate f_s yields the spectral representation shown in Fig. 1. The analytic discrete-time signal $z(n)$ has the spectrum shown in Fig. 2. Consider now the signals $s'(n) = s(n/2)$ and $z'(n) = z(n/2)$ sampled at the same frequency f_s. Their spectra are as shown by Figs. 3 and 4. The maximum spectral component of the signal $s'(n) = s(n/2)$ is now $2f_m = f_s$, and the periodicity of the spectrum is still f_s unchanged. *Aliasing occurs* due to the overlap between positive and negative frequencies (Fig. 3).

The maximum spectral component of the signal $z'(n) = z(n/2)$ is $2f_m = f_s$ and the periodicity of the spectrum is f_s. *No aliasing occurs* because the power density spectrum is zero for negative frequencies and because a single complex sample of the analytic signal is as effective as two real samples of the real signal in eliminating aliasing.

Example: To emphasize this point, let us study the spectral extent of the WD of a signal with the spectrum in Fig. 5 (the analytic signal is shown in shading). In the frequency domain, the WD and WVD can be expressed symmetrically as follows:

$$\rho(t, f) = \int_{-\infty}^{+\infty} S(f + m/2) \cdot S^*(f - m/2) \cdot e^{j2\pi mt} \cdot dm \qquad (11)$$

$$W(t, f) = \int_{-\infty}^{+\infty} Z(f + m/2) \cdot Z^*(f - m/2) \cdot e^{j2\pi mt} \cdot dm \qquad (12)$$

where $S(f)$ is the spectrum of $s(t)$, and $Z(f)$ is the spectrum of $z(t)$.

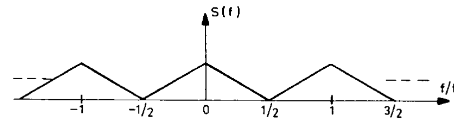

Fig. 1. An example of spectrum of a real discrete-time signal $s(n)$, sampled at the Nyquist rate fs.

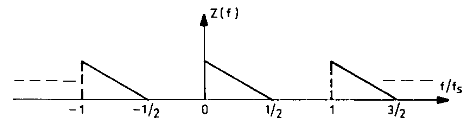

Fig. 2. Spectrum of the analytic discrete-time signal $z(n)$ associated with the real signal $s(n)$ shown in Fig. 1.

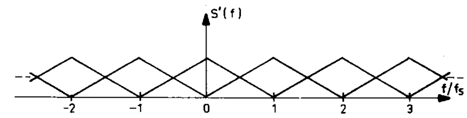

Fig. 3. Illustration of the consequence of sampling the real signal $s(n/2)$, at the same frequency fs as the signal in Fig. 1. *Aliasing occurs.*

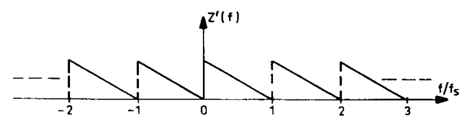

Fig. 4. Illustration of the consequence of sampling the analytic signal $z(n/2)$, at the same frequency fs as the analytic signal in Fig. 2. *Aliasing does not occur*, because there is no overlap between positive and negative frequencies.

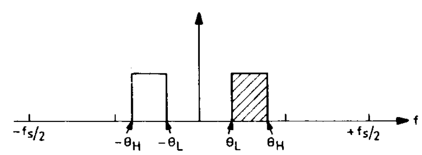

Fig. 5. An example of spectrum of a bandpass real signal. The corresponding analytic signal has the spectrum shown in shading.

Taking into account the periodicity of the spectrum of the sampled signal $s(n)$, (Figs. 3 and 4), we obtain the following spectrum for the WD and WVD (Fig. 6) (WVD is shown in shading).

Therefore, the WVD is straightforward to compute, without any particular precautions. It suffices to compute the DFT of the product $z(n + m/2) \cdot z^*(n - m/2)$ and to scale the frequency axis [2] [see (10)]. $z(n)$ is calculated either in the time domain using an FIR filter or in the frequency domain as follows:

1) take the N-point DFT of $s(n)$ to give $S(k)$, $K = 0, N - 1$;
2) define

$$Z(k) = \begin{cases} S(k), & k = 0 \\ 2 \cdot S(k), & k = 1, N/2 - 1 \\ 0, & \text{otherwise;} \end{cases} \qquad (13)$$

3) calculate $z(n) = \text{IDFT}\,(Z(k))$ where IDFT = inverse DFT.

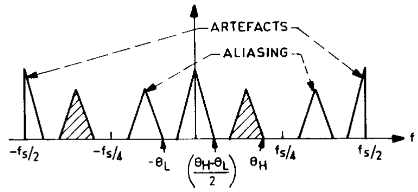

Fig. 6. Spectral spread of the Wigner–Ville distribution (WVD) and the Wigner Distribution (WD) of the signal of Fig. 5. The WVD corresponds to the shaded area. Artifacts and aliasing occur with the WD.

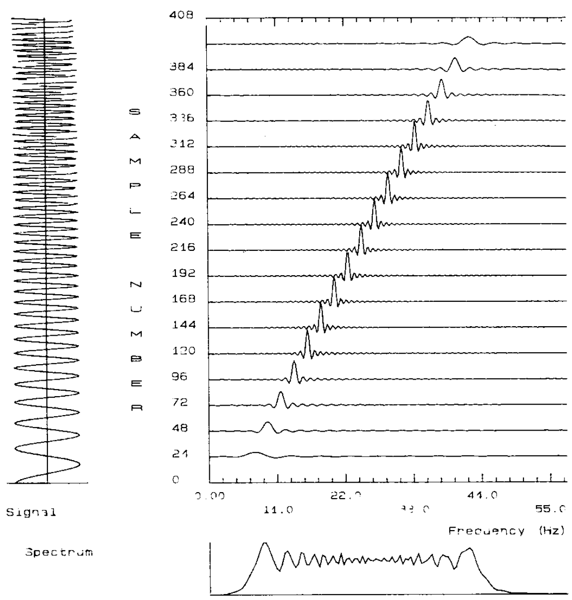

Fig. 7. Wigner-Ville Distribution (WVD) of a chirp signal (linear frequency modulation). This time-frequency representation shows a concentration of energy along the instantaneous frequency of the signal, which corresponds to the FM law of the signal.

V. Symmetry and Reduction of Computation

As shown previously, the discrete WVD is basically a sequence of DFT's, each evaluated at the time of interest n, of the sequence $R(n, m) = z(n + m) \cdot z^*(n - m)$.

We have: $R(n, -m) = R^*(n, m)$ [15].

A Fortran program has been written which exploits this symmetry by doing the following:

1) calculating $R(n, m)$ for $m > 0$ over the windowed range; and

2) using special FFT programs which exploit the symmetry of $R(n, m)$. The program halves the number of arithmetic operations as well as the amount of storage needed. All the calculations are executed within the same array [15].

The algorithm described above has been implemented on a VAX-11/750, running UNIX. It is based on the APL program implemented in 1978 for ELF-Aquitaine Geophysical Research Center, Pau, France [2].

VI. Computation of Other Time-Frequency Distributions (TFD's)

It was shown that we can define a general formulation of TFD's as follows:

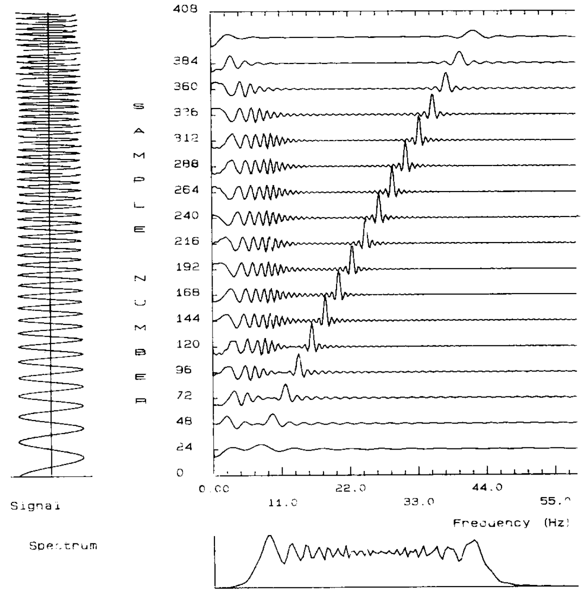

Fig. 8. Wigner Distribution of a chirp signal (linear frequency modulation). This representation exhibits in addition low-frequency artifacts, and should not be used for time-frequency signal analysis.

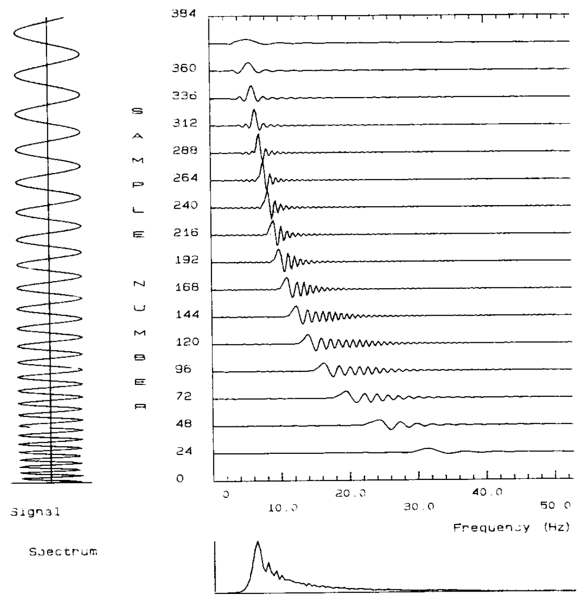

Fig. 9. Same as Fig. 7 with a hyperbolic FM signal.

$$\rho(t, f) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{+\infty}^{+\infty} e^{j2\pi n(u-t)} \cdot w(n, \tau) \cdot z\left(u + \frac{\tau}{2}\right) \cdot z^*\left(u - \frac{\tau}{2}\right) \cdot e^{-j2\pi f\tau} \cdot dn \cdot du \cdot d\tau, \tag{14}$$

where $w(n, \tau)$ is a weighting function that belongs to the Cohen's class (for more details, see [16]). An appropriate choice of $w(n, \tau)$ yields one of the previously proposed definitions of TFD's [16].

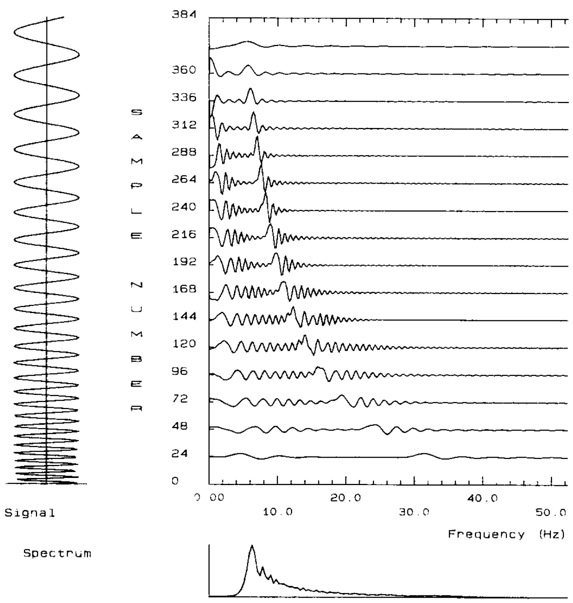

Fig. 10. Same as Fig. 8 with a hyperbolic FM signal. The WD exhibits low-frequency artifacts, which distort the time-frequency representation of the signal. It should not be used for time-frequency signal analysis.

$\rho(t, f)$ is related to $W(t, f)$ as follows:

$$\rho(t, f) = W(t, f) \underset{(t, f)}{*} \tilde{w}(t, f) \tag{15}$$

where $\tilde{w}(t, f)$ is the 2-D Fourier transform of $w(n, \tau)$.

We can therefore compute any other time-frequency distribution by applying the proper "time-frequency" transformation to the result of the program WVD.

VII. Conclusion

The Wigner–Ville Distribution is the version to be used for a better time-frequency analysis. It is straightforward to compute, and no aliasing is created by the method. It contains essentially the same information as the Wigner Distribution, but does not exhibit low-frequency artifacts produced in the real WD. The WD could be justified *only if* one already knows the signal to be analyzed and knows where the artifacts produced lie. Even, in this case, the WD must be computed at twice the rate of the WVD.

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The Use of Modulated Splines for the Reconstruction of Band-Limited Signals

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***Abstract*—A class of kernels of short duration is developed for interpolating band-limited signals. The class consists of modulated splines, and it utilizes the rapid attenuation of their Fourier transform. The design is based on a minimax MS error criterion.**

I. Introduction

A spline $m_n(t)$ of order n is a convolution of n identical rectangular pulses $p(t)$ of unit duration. Splines are used in interpolation problems [1]-[3] primarily because of their smoothness properties. They consist of n pieces of polynomials of degree $n - 1$, and they are continuous with continuous derivatives of order up to $n - 2$. This follows from the fact that their transforms tend to zero as $1/\omega^n$ as $n \to \infty$ [4]:

$$m_n(t) = p(t) * \cdots * p(t) \leftrightarrow M_n(\omega) = \left(\frac{\sin \omega/2}{\omega/2}\right)^n. \tag{1}$$

In this correspondence, we utilize their frequency domain properties to design a kernel $w(t)$ of the form

$$w(t) = \sum_{\nu=0}^{L-1} a_n s_n(t) \cos \frac{\pi \nu}{N} t \leftrightarrow W(\omega)$$

where

$$s_n(t) = m_n\left(\frac{nt}{2N}\right). \tag{2}$$

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