Continuum diffusion on networks: Trees with hyperbranched trunks and fractal branches

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The probability that a random walker returns to its origin for large times scales as \( t^{-\tilde{d}/2} \), where \( \tilde{d} \) is the spectral dimension. We calculate \( \tilde{d} \) for a class of tree structures using a renormalization technique on an infinite continued fraction. We consider a wide range of homogeneous networks based on replacing the branches of a self-similar tree with arbitrary fractals and composite fractals. We also consider a new class of inhomogeneous hyperbranched trees.

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I. INTRODUCTION

A number of physical [1], chemical [2], and biological [3] problems can be investigated through the study of random walkers on networks. An important parameter of a random walk on an infinite network is the spectral dimension \( \tilde{d} \) [4], which gives the long time scaling behavior of the probability that a walker returns to its origin \( C_0(t) \sim t^{-\tilde{d}/2} \). In addition to being an important parameter for characterizing diffusive transport, the spectral dimension is also directly linked to other physical properties of networks [5]. Because of the difficulty of analyzing diffusion on disordered structures, deterministic models, for which exact calculations can be performed, have been studied in great detail [6,7]. The properties of real disordered systems are modeled by studying networks which match measured structural characteristics such as the fractal dimension \( d_f \) [8], coordination number, and branching and looping behavior. This allows prediction of \( \tilde{d} \) and hence its dependence on structural factors. In this paper we derive \( \tilde{d} \) for a wide range of networks by generalizing a class of fractal trees to include hyperbranched trunks and fractal branches.

If a network can be embedded in \( d \) dimensional Euclidean space, it is said to be homogeneous and \( \tilde{d} \) is related to the fractal dimension \( d_f \) and the random walk dimension \( d_w \) by \( \tilde{d} = 2d_f/d_w \) [6]. For inhomogeneous networks \( d_f \) is not defined, and it is not guaranteed that the spectral dimension \( \tilde{d} \) exists [9]. Finding \( d_w \) and \( \tilde{d} \) for homogeneous networks can be difficult. For finitely ramified deterministic fractals, \( d_w \) can be derived using renormalization principles, but for other classes of fractals, \( d_w \) often needs to be numerically evaluated through the scaling of the second spatial moment of displacement with time \( \langle x^2 \rangle \sim t^{d_w} \) [10]. Given the key role that \( d_w \) and \( \tilde{d} \) play in the study of inhomogeneous and homogeneous networks [6,7], their analytic calculation remains a problem of ongoing interest [7,10,11].

For the spectral dimension, there has been significant progress in deriving analytic theorems [12] and exact results for “bundled structures” [13] and “fractal trees” [14–16], structures that are important in physical modeling [13]. Given that a wide variety of fractal structures can occur in nature, it is important to have available as general a class of models as possible. The aforementioned models do not incorporate branches with loops at all scales or hyperbranching, and these features may be important for modeling the properties of polymers [17]. In addition, prior results have been derived for a discrete random walker on structures with branches of integer length. Since certain network problems are more naturally formulated in the continuum (for example, see Refs. [18–20]), it would be useful to derive methods for modeling continuous-time–continuous-space random walkers.

In this paper we present a framework for calculating the spectral dimension of new classes of homogeneous and inhomogeneous structures. The networks incorporate fractal branches, which can contain loops at all scales, as well as hyperbranching in the main trunk. We derive all the results using the continuous-time–continuous-space diffusion equation which allows for noninteger branch lengths. The results can be specialized to discrete-time–discrete-space walkers through simple transformations [21].

In Figs. 1 and 2 we show a schematic of the classes of homogeneous tree fractals we analyze. If the branches of the tree are linear, the structures have been called nice trees of dimension \( D \) [14] [Fig. 2(a)]. We derive the spectral dimension for three generalizations of these structures; the case where each branch is a fractal [Fig. 2(b)], where each branch is a composite fractal [Fig. 2(c)] and a third case, where each branch is replaced by an arbitrary finite subnetwork [Fig. 2(d)]. In Fig. 3, we provide an example of the type of inhomogeneous networks that we study in Sec. V.

II. DIFFUSION ON NETWORKS

The probability \( C(x,t) \) that a continuous-time–continuous-space random walker is at a point \( x \) at time \( t \) on a pipe is governed by the diffusion equation

![FIG. 1. A Bethe type lattice. We consider networks constructed by replacing the branches of the lattice with a range of different elements. Examples are shown in Fig. 2.](image-url)
where

\begin{align}
\frac{d}{H20849} \frac{d}{H20850} C(x,t) = \frac{\partial}{\partial t} C(x,t).
\end{align}

In order to formulate the equations governing the probability

of the initial condition;

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of nodes of a third copy. The spectral dimension of the network

created by infinitely extending this idea is given in Sec. V.

Taking Laplace transforms of Eq. (1) and applying the boundary conditions gives

\(C(x,t) = C_0(t), \quad C(b,t) = C_1(t), \quad C(0,0) = 0.\)

Taking Laplace transforms of Eq. (1) and applying the boundary conditions gives

\[c(x,s) = c_1(s) \sinh(x/\sqrt{D}) + c_0(s) \sinh(b-x)/\sqrt{D},\]

where

\[c(x,s) = \mathcal{L}[C(x,t)] = \int_0^\infty C(x,t) e^{-st} dt.\]

The flux entering the pipe at the node \(x=0\) is \(J_0\)

\(= -D \frac{\partial}{\partial x} C(x,t)|_{x=0}\) and the flux entering the pipe at \(x=b\) is \(J_1 = D \frac{\partial}{\partial x} C(x,t)|_{x=b}.\) For networks it is useful to express the fluxes in terms of the concentrations \(c_i\) at either end. This gives rise to the equations \(\mu \mathcal{A} = \mathcal{E}c.\) Here \(j_i(s) = \mathcal{L}[J_i(t)]\) are the Laplace transforms of the fluxes, \(\mu = \tanh(b\sqrt{D})/\sqrt{D}\) and

\[E = \begin{bmatrix}
1 & -\text{sech}(b\sqrt{D}) \\
-\text{sech}(b\sqrt{D}) & 1
\end{bmatrix}.\]  

We call \(E\) a flux-concentration matrix and for simplicity we now set \(D\) and \(b\) to unity. In the remainder of the paper we almost exclusively employ Laplace transformed functions. Rather than explicitly stating this dependence, we henceforth assume this relationship implicitly. Thus “concentration” will refer to the Laplace transform of the concentration, etc.

Using these relations, diffusion on an arbitrary network can be formulated as a system of algebraic equations. To link two or more pipes at a node we set the concentrations to be equal and take the total flux at the node to be zero (mass conservation). The \(j_i\) are now taken to represent the total flux entering node \(i\) [21]. For mass conservation \(j_i=0\), but if we take an instantaneous unit source at node \(i\), then the flux entering the node is \(J_i=g_i(t)\) implying that \(j_i = \mathcal{L}(J_i)=1\). Examples are shown in Eq. (B1) and Ref. [21].

The inverse of Eq. (2) can be written in the following two useful forms:

\[\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{bmatrix} \begin{bmatrix} j_0 \\ j_1 \end{bmatrix} = \begin{bmatrix} 1 & f \\ f & 1 \end{bmatrix} \begin{bmatrix} j_0 \\ j_1 \end{bmatrix}.\]

Here \(g_{ij}\) are Green’s functions and \(f = g_{01}/g_{00}\) is a first passage time. Their precise definitions are provided in Appendix A. In the second form we have used the fact that \(g_{11}=g_{00}\) and \(g_{01} = g_{10}\). While this is obvious for a single pipe, it actually holds for any symmetric network (see Sec. VI). Of course \(g_{ij}\) and \(f\) are actually the Laplace transforms of their underlying functions.

The second form provided in Eq. (3) can also be written as

\[g_{00}(1-f^2) \begin{bmatrix} j_0 \\ j_1 \end{bmatrix} = \begin{bmatrix} 1 & -f \\ -f & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix},\]

which allows a direct comparison with Eq. (2). This shows that \(\text{sech}(\sqrt{D})\) is a first passage time and \(\mu = g_{00}(1-f^2)\). Both forms of the flux-concentration equations (3) and (4) are employed throughout the paper.

A. Two illustrative examples

Consider connecting three pipes of different lengths end to end. The system of equations for the pipes \((i=0,1,2)\) is

\[\begin{bmatrix} c_0^{(i)} \\ c_1^{(i)} \end{bmatrix} = g_{00}^{(i)} \begin{bmatrix} 1 & f^{(i)} \\ f^{(i)} & 1 \end{bmatrix} \begin{bmatrix} c_0^{(i)} \\ c_1^{(i)} \end{bmatrix},\]

where \(g_{00}^{(i)}\) and \(f^{(i)}\) are the respective functions for the \(i\)th pipe. If we attach the right end node of the zeroth pipe to the
left end node of the first pipe we must have \( f_0^{(0)} + j_0^{(1)} = 0 \) and \( c_1^{(0)} = c_0^{(0)} \). At the junction of the first and second pipe we have \( f_1^{(0)} + j_0^{(2)} = 0 \) and \( c_0^{(2)} = c_1^{(1)} \).

For our example, taking the boundary conditions \( f_0^{(0)} = 1 \) and \( j_0^{(2)} = 0 \) corresponds to the case of a walker released at the left end node of pipe 0 and a reflective boundary condition at the right end node of pipe 2. Note that there are six equations: two boundary conditions, two continuity, and two mass conservation conditions. These fully determine the six unknown fluxes and concentrations within the system. In particular, we are interested in finding \( c_0(t) \) \( [c_0(s)] \) which we obtain through an iterative procedure.

Taking \( f_0^{(2)} = 0 \) for the second pipe gives \( f_0^{(2)} = f_0^{(2)} / g_0^{(2)} \). We write this relationship as \( j_0^{(2)} = h_0 = g_0^{(2)} \), with \( h_2 = 1 / g_0^{(2)} \). We now set \( f_1^{(1)} = -h_0 c_1^{(1)} \), where we have used the fact that \( f_1^{(1)} + j_0^{(2)} = 0 \) and \( c_0^{(2)} = c_1^{(1)} \). This leads to the system of equations:

\[
\begin{bmatrix}
  c_0^{(1)} \\
  c_1^{(1)} \\
\end{bmatrix}
= g_0^{(1)}
\begin{bmatrix}
  f_0^{(1)} \\
  f_1^{(1)} \\
\end{bmatrix}
= -h_0 c_1^{(1)}.
\]

Using the second of these equations, we can express \( c_1^{(1)} \) in terms of \( f_0^{(1)} \) to give \( c_1^{(1)} = (1 + g_0^{(1)} h_2) f_0^{(1)} / j_0^{(1)} \). Substituting \( c_1^{(1)} \) into the first row of Eq. (6) gives

\[
c_0^{(1)} = g_0^{(0)} [1 - g_0^{(0)} f_0^{(1)}] h_2 (1 + g_0^{(1)} h_2) - j_0^{(1)}.
\]

This equation can be expressed as \( j_0^{(1)} = h_0 c_0^{(1)} \), where \( h_1 \) depends on \( h_2 \) but not on \( f_0^{(1)} \) or \( j_0^{(1)} \) \( (i = 1, 2) \). In a similar way, we can write \( j_0^{(0)} = h_0 c_0^{(0)} \), where \( h_0 \) depends only on \( h_2 \). To determine \( c_0^{(0)} \), we apply the boundary condition \( j_0^{(0)} = 1 \) which gives \( c_0^{(0)} = (h_0) h_2 \). The remaining variables \( c_i^{(1)} \) and \( j_i^{(1)} \) \( (i \geq 1) \) can be found by backward substitution. Here, our goal is to find \( c_0^{(0)} \), so \( c_1^{(1)} \) and \( j_1^{(1)} \) \( (i \geq 1) \) do not need to be explicitly determined.

As a second example, we consider attaching two identical pipes, “1” and “1a,” to the end of pipe “0.” The equations for the three pipes are identical to Eq. (5) with labels \( i = 0, 1, 1a \). By analogy with the previous problem, we have the continuity conditions \( c_0^{(0)} = c_0^{(1)} = c_0^{(1a)} \) the mass conservation condition \( f_0^{(0)} + j_0^{(1)} + j_0^{(1a)} = 0 \), and we take the boundary conditions to be \( j_0^{(0)} = j_0^{(1)} = 0 \). If we set \( j_0^{(1)} = h_1 c_0^{(1)} \) and \( j_0^{(1a)} = h_1 c_0^{(1)} \), we can reduce the problem to the two by two system.

\[
\begin{bmatrix}
  c_0^{(0)} \\
  c_0^{(1)} \\
\end{bmatrix}
= g_0^{(0)}
\begin{bmatrix}
  f_0^{(0)} \\
  f_1^{(0)} \\
\end{bmatrix}
= -(h_2 + h_1 c_0^{(1)}).
\]

Since pipe 1 and 1a are identical, \( h_1 = h_1a \), allowing us to write \( j_0^{(0)} = -u h_1 c_0^{(1)} \) where \( u = 1 \) is the number of identical pipes attached at the right end node of network 0 \( (u = 2 \) above). To determine \( c_0^{(0)} \), the solution procedure is essentially identical to that used in the first example.

**B. Connection of n pipes**

We now consider an arbitrary number of elements where the pipes of the \( n \)th generation are denoted by \( \ell_a \). To analyze Bethe type networks, we form the \( (n+1) \)th generation by connecting \( u \) pipes to each pipe \( \ell_a \) at the node \( n+1 \), as described in the second example given in Sec. II A. The case \( u = 2 \) is shown in Fig. 1. We no longer need to label or explicitly consider the side branches; node \( n \) and pipe \( n \) will refer to the obvious locations in the main branch. We distinguish local and global quantities as follows: \( s_0^{(0)} \) and \( f_0^{(0)} \) are the functions \( g_0^{(0)} \) and \( f_0^{(0)} \), respectively, for the \( n \)th pipe. Likewise \( j_0^{(n)} \) is the flux at node \( n \) of the \( n \)th pipe. If no superscript is used, then the quantity corresponds to the entire network, i.e., \( c_n \) refers to the concentration at node \( n \) of the entire network.

Using this new notation, Eq. (5) for the \( n \)th pipe is

\[
g_0^{(n)}
\begin{bmatrix}
  1 & f_0^{(n)} \\
  0 & -1 \\
\end{bmatrix}
\begin{bmatrix}
  f_0^{(n)} \\
  c_0^{(n)} \\
\end{bmatrix}
= c_n^{(n+1)}.
\]

Observe, from the examples provided in Sec. II A, that we can set \( j_0^{(n)} = -u h_{n+1} c_n^{(n+1)} \) and \( j_0^{(n+a)} = h_a c_n^{(n)} \). It was implicitly proved that \( j_0^{(n)} = h_a c_n^{(n)} \) for any terminating network and hence, following the argument in Sec. II A, the mass conservation condition implies \( j_0^{(n)} = -u h_{n+1} c_n^{(n+1)} \). This transformation appears to allow \( c_n^{(n+1)} = 0 \) as a solution to Eq. (7). We ignore this unphysical case and require \( c_n^{(n+1)} > 0 \).

The equations are then solved to determine \( c_n^{(n+1)} \) in terms of \( h_{n+1} \).

\[
(g_0^{(1)} h_{n+1})^{-1} = 1 - (f_0^{(n)})^2 \left[ 1 + (g_0^{(n)} h_{n+1})^{-1} \right].
\]

For reasons that will become clear in the next section, we rewrite Eq. (8) by multiplying the term \( (g_0^{(n)}) h_{n+1} \) to the right-hand side (RHS) of Eq. (8) by \( s_0^{(n+1)} / g_0^{(n+1)} \);

\[
(s_0^{(n+1)} h_{n+1})^{-1} = 1 - (f_0^{(n)})^2 \left[ 1 + \frac{s_0^{(n+1)}}{g_0^{(n+1)}} \right] \]

Either of Eqs. (8) or (9) allow all the \( h_i \) to be determined for a finite network. The no-flux boundary condition is incorporated into \( h_{n+1} \), where \( n+1 \) is the final node of the network. \( c_0 \) can then be found from \( c_0 = 1 / h_0 \). The functions \( 1 / h_n \) are the return to origin Green’s functions for the network that lies to the right of node \( n \).

For a discrete random walker, it is well known that the probability generating functions associated with random walk quantities on tree lattices can be expressed in terms of a continued fraction [22]. To our knowledge, this is the first time the infinite continued fraction for a continuous-time-continuous space random walker has been formulated.

**III. PRIOR RESULTS**

It is useful to show how the continued fraction can be used to derive earlier results. The ideas will be used in subsequent sections. We first find \( c_0 \) for the infinite Bethe lattice [23,24] with co-ordination number \( u+1 \). The relevant system for the problem is

\[
\frac{\coth(\sqrt{s})}{\sqrt{s}}
\begin{bmatrix}
  1 & \operatorname{sech}(\sqrt{s}) \\
  \operatorname{sech}(\sqrt{s}) & 1 \\
\end{bmatrix}
\begin{bmatrix}
  f_0^{(n)} \\
  j_0^{(n)} \\
\end{bmatrix}
= c_n^{(n+1)}.
\]

For simplicity, we have used explicit expressions for \( g_0^{(n)} \) and \( f_0^{(n)} \). A recurrence relationship is obtained by taking \( f_0^{(n)} = -u h_{n+1} c_n^{(n+1)} \) and \( j_0^{(n)} = h_a c_n^{(n)} \) in Eq. (10) and solving for \( h_n \) to get
If we consider infinitely many pipes \((n \to \infty)\), then the continued fraction given in Eq. (11) becomes an infinite periodic fraction. Replacing \(h_n\) and \(h_{n+1}\) by \(h_0\) and solving the resultant quadratic gives

\[
c_0 = \frac{1}{h_0} = \frac{2u \tanh(\sqrt{s})s^{-1/2}}{(u-1) + \sqrt{(u-1)^2 + 4u \tanh(\sqrt{s})^2}}.
\]

\(c_0\) must be multiplied by \(u+1\) to model a full Bethe lattice as we have only considered one of \(u+1\) branches. The expression is equivalent to prior results [23, 24] if the continuum-discrete transformation [21] is applied.

We now consider classes of networks obtained by adding a branch at every node \(i \geq 1\) of the basic Bethe lattice. The attachment of a branch \(B\) implies the total flux entering the \(n\)th pipe at node \(n+1\) is now the addition of two fluxes; the flux from the \((n+1)\)th pipe and the flux from the branch. We therefore take \(j_1^n = -uh_{n+1}c_{n+1}h_B^n\) and \(j_0^n = h_n c_n\) in Eq. (7) to derive a recurrence relationship for the modified network:

\[
(g_0^{(n)} h_n)^{-1} = 1 - (f^{(n)})^2 \times \{1 + [g_0^{(n)}(uh_{n+1} + h_B^n)]^{-1}\}^{-1}.
\]

(12)

Here, \(h_B^n\) is associated with the branch attached to the \(n\)th node of the network.

Consider an infinite pipe \((u=1)\) composed of finite pipes \(n\) having identical lengths with an identical branch \(B\) \((h_B^n = h_B)\) attached at every node. In this case, the recurrence relationship (12) reduces to

\[
h_n = \sqrt{s} \left( \coth(\sqrt{s}) - \frac{\csc(\sqrt{s})^2}{h_{n+1} + h_B + \sqrt{s} \coth(\sqrt{s})} \right),
\]

which can be solved, since it is a periodic infinite fraction, to give

\[
h_0 = \frac{1}{2} \left( -h_B + \sqrt{(h_B)^2 + 4 \sqrt{s} \coth(\sqrt{s}) h_B + 4s} \right).
\]

(13)

As an example, the function \(h_B^n\) for a two-dimensional (2D) lattice can be found [21] from its discrete counterpart [26] giving \(h_B^n = \tanh(\sqrt{s})K[s\csc(\sqrt{s})^2/(2\pi \sqrt{s})]\), where \(K(x)\) is the complete Elliptic integral of the first kind [27]. Using this in Eq. (13), gives \(c_0\) for a semi-infinite kebab lattice as shown in Fig. 4 [25]. Likewise \(c_0\) for a semi-infinite 2D comb lattice is taken by attaching two infinite pipes at every node, implying that \(h_B^n = 2 \sqrt{s}\). One can construct \(c_0\) for a \(d\)-dimensional comb [28] through similar arguments. All of these examples are cases of bundled structures whose spectral dimensions were derived in [13].

**IV. SPECTRAL DIMENSION OF NT_D TREES**

Although Eq. (9) provides a formal solution for \(h_0\), the analysis of the infinite continued fraction is greatly simplified if the pipes of generation \(\ell\), scale in a particular way. In this section we focus on the \(n\)th pipe having length \(2^{\ell n}\), where \(n > 0\). For the examples in the previous section \(n = 0\). For \(n > 0\) and integer, the resultant networks have been called \(NT_D\) trees [15]. Here we show how to renormalize the continued fraction to obtain the spectral dimension for these networks.

Since every pipe has length \(2^{\ell n}\), \(f^{(n)} = \sech(2^{\ell n} \sqrt{s})\) and \(g_0^{(n)} = \coth(2^{\ell n} \sqrt{s}) / \sqrt{s}\) [this follows from Eqs. (2) and (3)]. Note that \(f^{(n)}\) and \(g_0^{(n)}\) have the following key properties:

\[
f^{(n)}(\rho(s)) = f^{(n+1)}(s),
\]

(14)

\[
g_0^{(n)}(\rho(s)) = g_0^{(n+1)}(s),
\]

(15)

where \(\rho(s) = 4s\). The function \(\rho(s)\) is introduced to allow the results to be extended to more general fractals later.

Now because \(g_0^{(n)} h_n\) given in Eq. (9) is a function of \(f^{(n)}\), \(g_0^{(n+1)} h_n, g_0^{(n+1)} h_{n+1}\), we can exploit the properties of Eqs. (14) and (15) to relate \(g_0^{(n)} h_n\) to \(g_0^{(n+1)} h_{n+1}\). In doing so, it can be shown that it is possible to transform the infinite continued fraction (9) for \(h_0\) to a finite continued fraction.

Setting \(s = \rho(s)\) in the RHS of Eq. (9) and using Eqs. (14) and (15) returns the RHS of Eq. (9) with \(n\) replaced by \(n + 1\) which is just \((g_0^{(n+1)} h_{n+1})^{-1}\). Therefore for an infinite network

\[
[g_0^{(n)}(\rho) h_n(\rho)]^{-1} = (g_0^{(n+1)} h_{n+1})^{-1}.
\]

(16)

For a finite network the continued fraction terminates and this equation does not hold.

When \(n = 0\), we find that

\[
g_0^{(0)}(\rho) h_0(\rho) = h_1^{-1}.
\]

(17)

This equation provides a second relationship between \(h_0\) and \(h_1\) which can be combined with the first step of the iterative process given in Eq. (9) to eliminate \(h_1\) and express \(h_0\) in terms of \(h_0(\rho)\).

From Eq. (9) we know that \(h_0\) is given by

\[
(g_0^{(0)} h_0)^{-1} = 1 - (f^{(0)})^2[1 + (g_0^{(0)} h_0)^{-1}]^{-1}.
\]

(18)

Substituting Eq. (17) into Eq. (18) gives

\[
(g_0^{(0)} h_0)^{-1} = 1 - (f^{(0)})^2 \left( 1 + \frac{g_0^{(0)} h_0(\rho)}{g_0^{(0)} h_0(\rho)} \right)^{-1}.
\]

This equation represents a renormalization of the infinite continued fraction for \(h_0\), and is the key to the results of this paper.
To finally relate the problem to $c_0$, we recall that $c_0 = 1/h_0$ and $g^{(a)}_{00}(t) = g^{(a)}_{ij}(t)$, which gives an implicit equation for $c_0$:
\[
c_0 = g^{(a)}_{00} - (g^{(a)}_{01})^2 \left( g^{(a)}_{00} + \frac{(3)_{I_0} c_0(\rho)}{u g^{(a)}_{00}(\rho)} \right)^{-1}.
\] (19)

A. Asymptotic analysis

From the expression (19) for $c_0(s)$, we can find the long time asymptotic behavior of $C_0(t)$ by analyzing $c_0(s)$ as $s \to 0$. If $C_0(t) \sim t^{-\delta}$ as $t \to \infty$, then, by standard Tauberian theorems [29,30], it can be shown that $c_0(s)$ has the following behavior as $s \to 0$:
\[
c_0 = \begin{cases} 
  k_i s^\delta + O(s^{\delta+1}) & -1 < \delta < 0 \\
  k_0 \delta \ln(s) + O(\delta^2 \ln(s)) & \delta = 0, 1, 2, \ldots ,
\end{cases}
\]
\[
k_0 + k_0 s^\delta + O(s^{\delta+1}) & \delta > 0, \delta \neq 0, 1, \ldots
\]
(20)
where $k_i$ are constants. Since $g^{(a)}_{ij}(s)$ are Green’s functions of finite structures, they must have the following series expansion:
\[
g^{(a)}_{ij}(s) = (s A_n)^{-1} + O(s^2),
\]
where $A_n$ is the mass of $\ell_n$. This is a direct consequence of the fact that as $t \to \infty$, the probability of being at the origin (or indeed any node) of the finite pipe is equal to the reciprocal of the mass of the pipe. Although length and mass are identical for a pipe, for the more general elements considered in later sections, the mass and length differ.

Using the asymptotic forms (20) and (21) in Eq. (19) and equating coefficients of the resultant expansion, we find that
\[
1 = \frac{4 e^{4\kappa_0}}{\kappa A}.
\]
for all possible cases of $\delta$, implying that $\delta = -1 + \ln(A_n)/\ln(4^n)$. In the above, $\kappa = 2$ denotes the fractional increase in the length of the pipe for each iteration.

Finally, in order to calculate the spectral dimension, it is clear that $\delta = \bar{d}/2 - 1$. Hence
\[
\bar{d} = 1 + \frac{2 \ln(u)}{v \ln(4)},
\]
(22)
which agrees with prior results [31] if $v$ is an integer. For the continuum case $v$ can be any positive number. Given that the calculation of the spectral dimension is not dependent on the chosen starting site within the infinite network [32], we only need to perform the asymptotic analysis at one node. In subsequent sections we apply the renormalization and scaling ideas to new classes of networks.

V. INHOMOGENEOUS NETWORKS

We now calculate $\bar{d}$ for an interesting class of inhomogeneous networks generated iteratively. Let $\beta_0$ denote an infinite network with origin at node 0 and spectral dimension $\bar{d}_0$. Now attach the origin of $\beta_0$ to node $i$ of a copy of $\beta_0$, and denote the new network as $\beta_1$ (the origin of $\beta_1$ is the same as the origin of $\beta_0$). Next, attach the origin of $\beta_1$ to node $i$ of a new copy of $\beta_0$ to form $\beta_2$ and so on. The resultant infinite network $\beta_\infty$ has no fractal dimension (i.e., it is inhomogeneous). As an elementary example, let $\beta_0$ be an infinite pipe and set $i=1$, in which case $\beta_\infty$ is a comb with infinite branches.

As in previous sections, our aim is to calculate $c_0 = 1/h_0$. The sequence $h_n$ is given by Eq. (8) with the exception of $h_{i-1}$ which is equal to
\[
(\delta_0^{(i-1)} h_{i-1})^{-1} = 1 - (f^{(i-1)})^2 \times \left\{ 1 + \left[ g^{(0)}_0 (u h_0 + h_0^B) \right]^{-1} \right\}^{-1}.
\]
(23)
The function $1/h_0^B$ is the Green’s function of the added network $B$. This network is equivalent to $\beta_\infty$ except that it has one branch ($\beta_0$) removed. The two structures are essentially equivalent, and therefore must share the same Green’s function, so $1/h_0^B = 1/h_0$. The function $1/h_i$ in Eq. (23) is the Green’s function of the remainder of the original network. Because it is just a rescaled version of $\beta_0$, its scaling behavior is identical to that of $\beta_0$ which can be obtained from Eq. (20) with $\delta = \bar{d}/2 - 1$. Equation (23) now gives $h_{i-1}$ in terms of a function of known scaling $h_1$ and the function $h_0^B (= h_0)$ which at this stage is unknown, but it is to be determined.

We now eliminate $h_1, h_2, \ldots, h_{i-2}$ from Eq. (8) to express $h_0$ as a function of $h_{i-1}$, and hence as a function of $h_1$ and $h_0^B (= h_0)$. It is clear that $h_0$ is a function of $h_1$, and $h_1$ is a function of $h_2$, etc. We can eliminate all intermediate $h_n$ (1 $\leq n < i-1$) by repeated substitution in the continued fraction to obtain a result of the form
\[
h_0 = \frac{1}{\alpha_i h_{i-1} + 1}.
\]
(24)
where $\alpha_i$ are functions of $s$. Using the fact that $c_0 = 1/h_0$ we have
\[
c_0 = \frac{\alpha_i h_{i-1} + 1}{\alpha_i h_{i-1} + 1}.
\]
(25)
To derive the scaling behavior of each $\alpha_i$ we consider three problems on finite networks associated with $\beta_0$. First, consider the truncated network $"P"$ created by terminating the network at node $i-1$ with a zero flux boundary condition ($h_{i-1}^0 = 0$). With $h_{i-1}^0 = 0$, Eq. (25) becomes $c_0^{(i-1)} = \alpha(s)$, which is now the Green’s function for the finite network $I$ with total mass $M_{i-1}$. Thus $\alpha_1 = 1/(M_{i-1}s) + O(s^3)$.

We now take a homogeneous Dirichlet condition at node $i-1$ of network $I$ ($c_0^{(i-1)} = 0$). This is equivalent to taking $h_{i-1}^0 \to \infty$, in which case Eq. (25) becomes $c_0^{(i-1)} = \alpha_1^0/\alpha_2^0$, where $c_0^{(i-1)}$ is the concentration at the origin of the modified problem. Since $c_0^{(i-1)} (t) \to 0$ as $t \to \infty$, $c_0^{(i-1)} (s) = \mathcal{L}[c_0^{(i-1)} (t)]$ is nonsingular at $s=0$, and therefore $c_0^{(i-1)} = O(s^3)$.

Finally, consider the truncated network $\Pi$ formed by terminating the network at node $i$ with a zero flux boundary condition ($h_i^0 = 0$). If we set $h_i^0 = 0$ in Eq. (8) then $h_{i-1}^0 = 1/8 h_0^{(i-1)}$. Rewriting Eq. (25) for network $\Pi$ gives
As an example, if a comb with \( u = 1 \), then the mass of the finite network \( H_{11009} \) is shown in Fig. 6. Here we show that the results of the previous sections can be simply modified for this case.

Consider a flux-concentration matrix for a fractal with \( N \) nodes of the form \( \mathbf{M} = \mathbf{E}_0 \mathbf{c} \), with \( \mathbf{f}^0 = (f_{00}, \ldots, f_{11}) \). If we ignore the first and last equations we have an undetermined system of \( N-2 \) equations for \( N \) variables. This can be rearranged to express the \( N-2 \) intermediate concentrations in terms of \( c_0 \) and \( c_1 \) (see Appendix B and [21]) for an example). After substitution, we find the \( N \times N \) equations are replaced by the \( 2 \times 2 \) system:

\[
\mathbf{E}_0 \begin{bmatrix} f_{00} \\ f_{11} \end{bmatrix} = \begin{bmatrix} E_{00} & E_{01} \\ E_{10} & E_{11} \end{bmatrix} \mathbf{c} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}.
\]

The reduction of \( E \) to a \( 2 \times 2 \) matrix \( \overline{E} \) can be performed for arbitrary networks with two exterior connections. \( \overline{E} \) is the flux-concentration matrix for an “effective reduced network.”

Re-writing the equations for the effective reduced networks as in Eq. (3) for iteration \( n \) gives

\[
\begin{bmatrix} c_n \\ c_{n+1} \end{bmatrix} = g_{00}(n) \begin{bmatrix} 1 \\ f_{01}(n) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix},
\]

where \( g_{00}(n) \) and \( f_{01}(n) \) are Green’s functions and first passage times associated with the fractal \( \ell_n \). Writing the matrix in the form shown in Eq. (27) requires \( g_{11} = g_{00} \) and \( g_{01} = g_{10} \). This restricts our results to fractals with end-to-end symmetry.

Since these equations have an identical form to Eq. (7) all of the results in Sec. IV can be extended if Eqs. (14) and (15) hold. Since \( f_{01}(n) \) is a first passage time of a finitely ramified deterministic fractal, there exists a \( \rho(s) \) such that \( f_{01}(n) = f_{01}(\rho(s)) \) [33,34]. An example is given in Appendix B. The second Eq. (15) also holds for any \( F \) (see Appendix C).

In order to apply the series expansions provided in Sec. IV A, we note that \( \rho(s) \) has the following Taylor series expansion:

\[
\rho(s) = a^s + O(s^2),
\]

where \( a \) is linked to the random walk dimension \( d_w \) of \( F \) such that \( a = L^{d_w} \). Here \( L \) is the rescaling of length from the \( i \)th to the \((i+1)\)th iteration of \( F \). The Sierpinski gasket, for example, has \( L = 2, d_w = \ln(5)/\ln(2) \) and \( a = 5 \). The pipe used in the \( NT_D \) (Sec. IV) tree can be viewed as a fractal with \( L = 2 \) and \( d_w = 2 \) which implies that \( a = 2^2 = 4 \). For the pipe, note that \( \rho(s) = 4s \) exactly. The appearance of the exponent \( v \) for \( v \geq 2 \) in the expansion of \( \rho(s) \) is explained in Appendix B.

The analysis presented in Sec. IV A holds for arbitrary \( \rho(s) \) and gives

![Diagram of a fractal network](image-url)
Similarly, if we define
that as that can relate all fractals. We take the subnetwork $\ell_n$ to be a fractal $\mathcal{F}(1)$ of iteration $v(1)n$. We now replace all of the pipes in the subnetwork by a fractal $\mathcal{F}(2)$ of iteration $v(2)n$. An example is given in Fig. 2(c): $\mathcal{F}(1)$ is a $T$ tree of iteration $n$ (i.e., $v(1)=1$) and $\mathcal{F}(2)$ is a Sierpinski gasket of iteration $n[v(2)=1].$

For the $n$th iteration the inverse flux-concentration equations of the effective reduced network are

$$\begin{bmatrix} c_n \\ c_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & v(s) \\ -1 & 1 \end{bmatrix} \begin{bmatrix} f^{(n)} \\ f^{(n)} \end{bmatrix}.$$

It can be shown that $f^{(n)} = f^{(1)}(f^{(2)}(s))$, where $f^{(1)}$ and $f^{(2)}$ are the first passage time functions for the fractals $\mathcal{F}(1)$ and $\mathcal{F}(2)$ that occur in the $n$th generation. An example of this relationship is given in Appendix B. Careful examination of the first passage time shows that the functional relationship satisfied by $f^{(1)}$ and $f^{(2)}$ is

$$f^{(1)}\left(f^{(2)}(s)\right) = f^{(n+1)}(s).$$

The combination of functions appearing in the argument of the left-hand side depends on $n$ and therefore no $\rho^*$ exists that can relate all $f^{(n)}$ to $f^{(n+1)}$ as in prior sections. However, an analogous result can be shown for small $s$, which is all that is needed to calculate the spectral dimension.

The expansion of the first passage time of the fractal $\mathcal{F}(2)$ is $f^{(2)} = 1 - \gamma_2 s + O(s^2)$ because the underlying first passage time is a probability density (so $f^{(2)}|_{s=0} = 1$). Using Eq. (14) and the expansion $\rho(s) = a_2^0 s + O(s^2)$ implies that $f^{(3)}(s) = f^{(2)}(\rho(s)) = 1 - \gamma_2 a_2^0 s + O(s^2)$.

Similarly $f^{(n)} = 1 - \gamma_2 a_2^{(n)} s$ for the fractal $\mathcal{F}(1)$. Combining these results gives

$$f^{(n)}(\rho(s)) = a_2^{(1)} a_2^{(n)} + O(s^2) = a_2^{(n+1)} s + O(s^2).$$

If we define $\rho^* = \rho(s)$, which has a Taylor series expansion of $\rho^* = a_2^{(1)} a_2^{(n)} s + O(s^2)$, then $f^{(n)}(\rho^*) = f^{(n+1)}(s) + O(s^2)$.

Now consider the Green’s function $g^{(n)}_0$ for $\ell_n$. We know that as $s \to 0$, $g^{(n)}_0 = 1/(A s) + O(s^0)$ and therefore $g^{(n+1)}_0 / g^{(n)}_0 = A_{n+1} / A_n + O(s) = A + O(s)$, where $A$ is the increase in mass from $\ell_n$ to $\ell_{n+1}$. Now since

$$\frac{g^{(n+1)}(s)}{g^{(n)}(s)} = \frac{g^{(n+2)}(s)}{g^{(n+1)}(s)} + O(s),$$

we rewrite our key equation

$$\bar{d} = 2 \frac{\ln(n)}{v} + \bar{d}_F.$$
The above method can be extended to multiple fractals. With the obvious extension of notation, the spectral dimension for $M$ fractals is
\[
\bar{d} = 2 \frac{\ln(Au)}{\ln \left( \prod_{i=1}^{M} a_i^{(i)} \right)} \quad \text{and} \quad A = \prod_{i=1}^{M} a_i^{(i)2}. \]

### VIII. SUBNETWORKS

The reason that we were able to find $\bar{d}$ in prior sections is because the first passage times and the Green’s function $g_{00}$ satisfied the renormalization equations (14) and (15). In this section we show how these equations can be used for general subnetworks (i.e., not necessarily fractal). First we examine the case where the basic pipe of the Bethe lattice (see Fig. 1) is replaced by an arbitrary finite subnetwork $\ell_n$, with two end nodes denoted by 0 and 1. We define $\ell_1$ to be identical to $\ell_0$ except that the internal pipes are replaced by pipes of length $2^n$ [see Fig. 2(d)], i.e., $\ell_1$ is a scaled up version of $\ell_0$.

Consider the flux-concentration matrix for a pipe in the subnetwork $\ell_n$, written in the form
\[
\xi_p = \begin{bmatrix} 1 - (f_p)^2 \\ f_p\end{bmatrix} = \begin{bmatrix} 1 & -f_p \\ -f_p & 1 \end{bmatrix} \begin{bmatrix} c_n \\ c_{n+1} \end{bmatrix}.
\]

Here $\xi_p$ and $f_p$ are the return to origin and first passage time functions for a pipe of length $2^n$. We use this notation to differentiate between the Green’s function of the pipe $\xi_p$ and of the subnetwork, which is denoted by $g_{00}$ as in earlier sections.

For an arbitrary subnetwork of $N$ nodes, the flux-concentration equations of $\ell_n$ will have the form
\[
\xi_p = \begin{bmatrix} 1 - (f_p)^2 \\ f_p\end{bmatrix} = E(f_p)c,
\]
where $E$ is an $N \times N$ matrix. All entries of $E$ are either constants or multiples of $f_p$ (for example, see Appendix B).

We now calculate the flux-concentration matrix $E$ for the effective reduced network by eliminating $N-2$ equations (as shown in the previous section and Appendix B) to give $\xi_p = (1-(f_p)^2)\bar{E}(f_p)c$. Inverting the $2 \times 2$ matrix $\bar{E}(f_p)$ gives $\xi_p = Q(f_p)d$, where $Q(f_p) = [1-(f_p)^2]^{-1}E^{-1}(f_p)$. The elements of $Q$, $q_{ij}(s)$, depend only on the first passage time $f_p$ of the pipe. Since $f_p(\rho(s)) = f_p^{(n+1)}(s)$ and $E$ (and therefore $\bar{E}$) are independent of $n$ it follows that $q_{ij}(\rho(s)) = q_{ij}^{(n+1)}(s)$. Given that $q_{ij}$ has this property, we can repeat the renormalization procedure demonstrated in Sec. IV.

First, the equivalent infinite continued fraction for the network composed of subnetworks $\ell_0, \ell_1, \ldots$, is found by taking $f_p^{(0)} = h_n c_n$ and $f_p^{(n)} = -h_{n+1} c_{n+1}$ in
\[
\xi_p = \begin{bmatrix} q_0 & q_{10} \\ q_{01} & q_{11} \end{bmatrix} \begin{bmatrix} j_{0} & j_{1} \\ j_{10} & j_{11} \end{bmatrix} = \begin{bmatrix} c_n \\ c_{n+1} \end{bmatrix},
\]
to get
\[
\xi_p = \begin{bmatrix} q_0 & q_{10} \\ q_{01} & q_{11} \end{bmatrix} \begin{bmatrix} j_{0} & j_{1} \\ j_{10} & j_{11} \end{bmatrix} = \begin{bmatrix} c_n \\ c_{n+1} \end{bmatrix}.
\]

### IX. CONCLUSION

We have calculated the spectral dimension for several classes of homogeneous and inhomogeneous networks through the renormalization of a continued fraction. These results expand the number of model structures with analytically known spectral dimensions. It was possible to derive the results for the fractal and composite fractal networks because of the interesting first passage renormalization function $\rho(s)$ [33,34]. In this paper we have shown how to exploit this function in a general formulation of diffusion on networks. Importantly, we were able to prove that certain ratios of Green’s functions could be renormalized using $\rho(s)$, allowing us to find a functional relationship between $c_0(\rho)$ and $c_0(\rho(s))$ for quite general structures.

The inhomogeneous trees considered in Sec. V appear to be new and may prove useful for modeling hyperbranched polymers [17]. It is interesting that the spectral dimension of the inhomogeneous trees have the same form as that obtained for bundled structures [13] even though they can be structurally quite different. In the final section we generalized the renormalization scheme to any finite subnetwork. Even though the growing subnetworks may contain loops of increasing size, the fact that $\bar{d}$ is not altered, indicates that there is insufficient coupling between the loops at different scales to fundamentally change diffusive transport.

An unusual aspect of this paper is that all of the analysis has been performed for a continuum random walker. This may prove directly useful for certain physical problems [19,20] which are more naturally formulated in the continuum. In addition, we believe that the analysis of diffusion on networks is simpler using the continuum formulation. In particular, the coordination number at each node, which fea-
tures strongly in analyses for discrete walkers does not appear. As a consequence, the flux-concentration matrix is symmetric and the series expansions of the Green’s functions are simpler. As we have established a direct correspondence between continuum and discrete walkers in an earlier paper [21], the continuum formulation may prove useful in future studies of discrete random walkers.

It is interesting that the continued fraction framework is able to provide an analytic route into so many different problems. Although we have only used this framework for structures with two end nodes, in future work [38] we show how it can be generalized to investigate branching trees made from interconnected branches with two or more starting nodes and two or more end nodes.

APPENDIX A: PROBABILISTIC INTERPRETATION

In this appendix we demonstrate the probabilistic interpretation of the elements of the matrices which arise in the flux-concentration equations. Solving the flux-concentration equations (2) of the pipe with \( j_2=1 \) and \( j_1=1 \) gives \( c^f=(g_{00},g_{10}) \). \( g_{00} \) is the Laplace transform of the probability that a walker released at site 0 at \( t=0 \) is at site 0 at later times on the closed pipe (i.e., it is a Green’s function). Similarly \( g_{10} \) is the Laplace transform of the probability that a walker released at site 0 at \( t=0 \) is at site 1 at subsequent times. Setting \( j=(0,1) \) gives \( g_{01} \) and \( g_{11} \). We can write the combination of both problems as \( \mu I=EG \), where the elements of \( G \) are \( g_{ij} \).

The relationship with first passage times is found by rewriting Eq. (2) in terms of Green’s functions and considering the problem

\[
\begin{bmatrix}
| g_{00} & g_{01} |
| g_{10} & g_{11} |
\end{bmatrix}
\begin{bmatrix}
| j_0 |
| 1 |
\end{bmatrix}
= \begin{bmatrix}
| 0 |
| c_1 |
\end{bmatrix}.
\]

If we solve this system, \( j_0 \) represents the Laplace transform of the flux at site 0 given that a source is released at site 1 at \( t=0 \). The first equation shows that \( j_0=g_{00}/g_{01} \). It is negative because \( j_0 \) is the flux entering at site 1. By definition, the flux into a trap is identical to the first passage time density function \( f \). Similarly \( -j_1=g_{10}/g_{11} \). The results are convolutions in time and can also be derived using random walker arguments [26].

APPENDIX B: DETERMINISTIC TREE

In this appendix we provide sample calculations for the deterministic tree which illustrate key ideas in the paper. The flux-concentration equations for the deterministic tree shown in Fig. 9(b) are \( \mu j=Ec(\sigma_0)c \), where

\[
E(\sigma_0) = \begin{bmatrix}
1 & 0 & -\sigma_0 & 0 & 0 \\
0 & 1 & -\sigma_0 & 0 & 0 \\
-\sigma_0 & -\sigma_0 & 4 & -\sigma_0 & -\sigma_0 \\
0 & 0 & -\sigma_0 & 1 & 0 \\
0 & 0 & -\sigma_0 & 0 & 1 \\
\end{bmatrix}
\]  
\tag{B1}

and \( \sigma_0=\text{sech}(\sqrt{\lambda}) \). To find the effective reduced network between nodes 0 and 1, we set \( j_2=j_3=j_4=0 \). The concentrations at the nodes 2, 3, and 4 can be expressed in terms of \( c_0 \) and \( c_1 \) as

\[
\begin{bmatrix}
| c_2 |
| c_3 |
| c_4 |
\end{bmatrix}
= \begin{bmatrix}
1/2 & \sigma_0 & 1 & 1 \\
2-\sigma_0/\sigma_0 & \sigma_0 & \sigma_0 & \sigma_0 \\
\end{bmatrix}
\begin{bmatrix}
| c_0 |
| c_1 |
\end{bmatrix}.
\]

Using these equations in Eq. (B1) gives

\[
\begin{bmatrix}
| j_0 |
| j_1 |
\end{bmatrix}
= 1/2 \begin{bmatrix}
4-3\sigma_0^2 & 1 & -\sigma_0^2 \\
2-\sigma_0^2 & 4-3\sigma_0^2 & 1 \\
\end{bmatrix}
\begin{bmatrix}
| c_0 |
| c_1 |
\end{bmatrix}.
\]  
\tag{B2}

These are the equations for the effective reduced network between nodes 0 and 1 of the tree. The above equations can be rewritten as

\[
g_{00}(1-\sigma_0^2)j_0 = \begin{bmatrix}
1 & -\sigma_0 \\
-\sigma_0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
| c_0 |
| c_1 |
\end{bmatrix},
\]  
\tag{B3}

where \( g_{00}=(4-3\sigma_0^2)\mu/[4(1-\sigma_0^2)] \) and \( \sigma_1=\sqrt{\lambda}/(4-3\sigma_0^2) \) is the first passage time of the reduced sub-network.

The next iteration of the tree is constructed from four copies of the zeroth iteration (see Fig. 9(a)). The full 17 \times 17 flux-concentration matrix for \( n=1 \) can be replaced by a 5 \times 5 matrix with the elements taken from the matrix of the effective reduced network equations (B2) or (B3) for the zeroth iteration. This gives

\[
\mu j_1 = 1/4 \begin{bmatrix}
4-3\sigma_0^2 & 4-3\sigma_0^2 \\
2-\sigma_0^2 & 2-\sigma_0^2 \\
\end{bmatrix}
E(\sigma_1)c.
\]  
\tag{B4}

where \( E(\sigma_1) \) is the matrix \( E(\sigma_0) \) shown in Eq. (B1) with \( \sigma_0 \) replaced by \( \sigma_1 \). For the next iteration we have

\[
\mu j_1 = 1/4 \begin{bmatrix}
4-3\sigma_1^2 & 4-3\sigma_1^2 \\
2-\sigma_1^2 & 2-\sigma_1^2 \\
\end{bmatrix}
E(\sigma_2)c.
\]  
\tag{B5}

This process can obviously be extended. Note that the renormalization procedure implies that \( \sigma_n(s)=\sigma_{n-1}(\rho(s))=\cdots=\rho(\hat{\rho}(\hat{\rho}(\cdots))v \text{ times})=\rho(s) \) for the deterministic tree \( \rho(s)=\text{arccsch}[\sigma_0^2/(4-3\sigma_0^2)]^2 \). As mentioned in the text the pipe can be interpreted as a fractal, in which case \( \rho(s)=4x \). In Sec. VI, we allow for the fractals in generation \( n \) of the network to be of iteration \( m_n \) (\( v \) an integer). From the above it is clear that \( \rho(s)=\hat{\rho}(\hat{\rho}(\cdots))v \text{ times} \hat{\rho} \) is the corresponding
function for $v=1$. If $\hat{\rho}=a s + O(s^2)$ then $\rho = a^v s + O(s^2)$. These expansions are required throughout the text.

In Sec. VII we consider fractal subnetworks where the pipes are constructed from a different fractal. The above example can be extended to this case as follows. Suppose we replace each pipe of the zeroth iteration of the deterministic tree by the zeroth iteration of the Sierpinski gasket. Equation (B3) for the gasket is

$$g_{00}(1 - \tau_1^2) = \left[\begin{array}{cc} 1 & -\tau_1 \\ -\tau_1 & 1 \end{array}\right] c,$$

where $\tau_1 = \tau_0 / (2 - \tau_0)$, $\tau_0 = \text{sech}(\sqrt{3})$, and $g_{00} = (\tau_0 - 2\mu / [2(\tau_0 - 1)(2 + \tau_0)])$. Now for the composite network equation, Eq. (B3) is replaced by

$$g_{00}^{\ast}(1 - \sigma_1(\tau_1)^2) = \left[\begin{array}{cc} 1 & -\sigma_1(\tau_1) \\ -\sigma_1(\tau_1) & 1 \end{array}\right] c,$$

where $g_{00}^{\ast} = g_{00}^\ast (4 - 3\tau_1^2) / 4$. The first passage time of the zeroth iteration of the composite lattice is therefore $f^{00}(s) = \sigma_1 (\tau_1(s))$.

**APPENDIX C: RATIO OF GREEN’S FUNCTIONS**

In this section, we show that the Green’s function $g_{00}^{(n)}$ for any finitely ramified deterministic symmetric fractal has the property (15) required for the renormalization argument. For the zeroth iteration of a fractal, the flux concentration equations are $\mu_j = E(\sigma_0) c$, where $E_j(\sigma_0)$ is a matrix with depends only on $\sigma_0$ [e.g., Eq. (B1)]. For the next iteration, the full flux-concentration equations can be reduced to the form of the zeroth iteration by employing reduced networks. The resultant equations can be written as

$$\mu_j = \gamma(\sigma_0) E(\sigma_1) c,$$

where $\gamma(\sigma_0)$ is a known function [see, e.g., Eq. (B4)]. In Appendix B, we showed that $\sigma_n = \sigma_{n-1} / (4 - 3\sigma_{n-1})$ for the deterministic tree. In general the form of $\sigma_n$ depends on the type of fractal. Repeating the procedure for higher iterations, the flux concentration equations can be expressed in the form [e.g., Eq. (B5)]

$$\mu_j = \prod_{i=0}^{n-1} \gamma(\sigma_i) E(\sigma_n) c.$$  \hspace{1cm} (C1)

Now $g_{00}$ is the first element of the Green’s function matrix,

$$g_{00}^{(n)} = \mu [E^{-1}(\sigma_n)]_{00} \prod_{i=0}^{n-1} \frac{1}{\gamma(\sigma_i)},$$

where $[E^{-1}(\sigma_n)]_{00}$ is the first element of $E^{-1}$. The key quotient can be written as

$$\frac{g_{00}^{(n+1)}}{g_{00}^{(n)}} = \frac{[E^{-1}(\sigma_{n+1})]_{00}}{[E^{-1}(\sigma_n)]_{00}} \gamma(\sigma_{n+1}),$$

and furthermore using $\sigma_n(s) = \sigma_{n-1}(\rho(s))$ we have

$$\frac{g_{00}^{(n+1)}(\rho(s))}{g_{00}^{(n)}(\rho(s))} = \frac{[E^{-1}(\sigma_{n+2})]_{00}}{[E^{-1}(\sigma_{n+1})]_{00}} \gamma(\sigma_{n+2}) \frac{g_{00}^{(n+2)}(s)}{g_{00}^{(n+1)}(s)}.$$  

Note that although the above proof has been given for $v=1$, it can be extended for any $v \geq 2$.

CONTINUUM DIFFUSION ON NETWORKS: TREES WITH...

[38] C. P. Haynes and A. P. Roberts (unpublished).