PROJECTIVE DIFFERENTIAL INVARIANTS OF TWO CURVES.

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INTRODUCTION.

The projective group of transformations in the plane

$X = \frac{a_1x + b_1y + c_1}{a_3x + b_3y + c_3}, \quad Y = \frac{a_2x + b_2y + c_2}{a_3x + b_3y + c_3},$

has eight essential parameters. According to the general theory,* in addition to the absolute invariants of a single curve, there are eight of “mixed” type, i.e., functions of both curves. Those of the first type have already been treated by G. H. Halphen in his celebrated thesis,§ where it is shown that the simplest are of orders 7 and 8. It follows that the eight mixed invariants are of orders not exceeding 6.

The object of the present paper is to determine them in algebraic form, leaving their geometric interpretation to be dealt with later.

1. THE AFFINE INVARIANTS.

In the first paper of this series† the six mixed invariants of the affine group were found, viz.,

$M_{21}, \quad M_{22}; \quad M_{31}, \quad M_{32}; \quad M_{41}, \quad M_{42}$

They were the solutions of a complete set of equations involving derivatives of the fourth order. The affine invariants of a single curve are

$J_0 J_4^{-3}, \quad J_0 J_4^{-2}$

where

$J_4 = 3y''y^v - 5y'''^2$

$J_0 = 9y''y^v - 45y'''y^{2v} + 40y'''^3$

$J_0 = 9y''y^{2v} - 63y'''y''y^v - 45y'''y^{2v^2} + 255y'''^2y^{3v} - 160y'''''$

---

† Affine Differential Invariants of Two Analytic Arcs.
This will be proved in a paper shortly to be published in this series, entitled "Affine invariant families of cuspidal cubics." In the meantime, their invariance can be readily verified. $J_4 = 0$ is the equation of all parabolas; $J_5 = 0$ is the equation of all conics; $J_6 = 0$ has no algebraic general solution, the attempt to integrate it leading to a Riccati equation of the so-called insoluble type.

The affine group being a sub-group of the projective, the invariants of the latter for two curves will be functions of

\[ M_{21}, M_{22}; M_{31}, M_{32}; M_{41}, M_{42}; \]

\[ J_{51}^2J_{41^{-3}}, J_{52}^2J_{42^{-3}}; J_{61}J_{41^{-2}}, J_{62}J_{42^{-2}} \]

the last four being invariants of the first type, that is, belonging to a single curve, indicated by the additional suffix 1 or 2.

2. Extended Infinitesimal Transformations of the Projective Group.

For the projective group we have, in addition to the extended infinitesimal transformations of the affine group, the following two which are non-affine:

\[
U_5 f = \Sigma x^2 \frac{\partial f}{\partial x} + \Sigma xy \frac{\partial f}{\partial y} + \Sigma (y - xy') \frac{\partial f}{\partial y'} - 3 \Sigma xy'' \frac{\partial f}{\partial y''} \\
- \Sigma (5xy'' + 3y') \frac{\partial f}{\partial y''} - \Sigma (7xy'' + 8y''') \frac{\partial f}{\partial y'''} \\
- \Sigma (9xy'' + 15y''') \frac{\partial f}{\partial y'''} - \Sigma (11xy''' + 24y''') \frac{\partial f}{\partial y''''} \\
U_6 f = \Sigma xy \frac{\partial f}{\partial x} + \Sigma y^2 \frac{\partial f}{\partial y} + \Sigma y (y - xy') \frac{\partial f}{\partial y'} - 3 \Sigma xy'y'' \frac{\partial f}{\partial y''} \\
- \Sigma (4xy'y''' + 3xy''^2 + 3y'y'' + yy''') \frac{\partial f}{\partial y'''} \\
- \Sigma (5xy'y'' + 10xy''y'' + 8y'y''' + 6y''^2 + 2yy''') \frac{\partial f}{\partial y''''} \\
- \Sigma (6xy'y''' + 15xy''y''' + 15y'y''' + 10y'''' + 30y'y''' + 3yy''') \frac{\partial f}{\partial y'''''} \\
- \Sigma (7xy'y''' + 21xy'y''' + 24y'y''' + 35xy''y''' + 60y'y''' + 40y'''' + 4yy''''') \frac{\partial f}{\partial y''''''}
The projective mixed invariants will be the solutions of

\[ U_1 f = U_2 f = U_3 f = U_4 f = U_5 f = U_6 f = U_7 f = U_8 f = 0. \]

The method of solution will be—in principle—to assume the 10 solutions (i) of the first six equations, and to use them as new variables in the last two. We shall thus obtain the projective invariants as functions of the affine. It will be seen that the work is not nearly as tedious as the equations might lead us to expect; but this is a common experience in group-theory.

3. Solutions of \( U_7 f = 0 \) to the Fourth Order.

We begin by finding the results of applying the operator \( U_7 \) to the relative invariants of the affine group. We find that

\[
\begin{align*}
U_7 (R_1) &= x_3 R_1; \quad U_7 (R_2) = x_1 R_2; \quad U_7 (R_3) = R_1 - x_2 R_3; \\
U_7 (R_4) &= -3x_1 R_4; \quad U_7 (R_5) = -3x_2 R_5;
\end{align*}
\]

\[
\begin{align*}
U_7 (R_6) &= (x_2 - 5x_1) R_6 - 3R_1 R_4; \quad U_7 (R_7) = (x_1 - 5x_2) R_7 - 3R_2 R_5 \\
U_7 (R_{10}) &= -10x_1 R_{10} + 6R_2 y_1'''; \quad U_7 (R_{11}) = -10x_2 R_{11} + 6R_5 y_2'''
\end{align*}
\]

From these we proceed to the affine absolute invariants:

\[
U_7 (M_{21}) = U_7 \left( \frac{R_2 R_4}{R_1 R_3^2} \right)
= M_{21} \left( x_2 - x_1 \right) - \frac{2R_2 R_4}{R_3^2}
= M_{21} \left( \frac{R_2 - R_4}{R_3} \right).
\]

Similarly,

\[
U_7 (M_{22}) = M_{22} \left( x_1 - x_2 \right) - \frac{2R_1 R_5}{R_3^2} = M_{22} \left( \frac{R_2 - R_4}{R_3} \right)
\]

\[
U_7 (M_{31}) = M_{31} \left( x_2 - x_1 \right) - \frac{6R_6}{R_4^2}
\]

\[
U_7 (M_{32}) = M_{32} \left( x_1 - x_2 \right) - \frac{6R_7}{R_5^2}
\]

\[
U_7 (M_{41}) = M_{41} \left( x_2 - x_1 \right) + \frac{6R_1 y_1'''}{R_4^2}
\]

\[
U_7 (M_{42}) = M_{42} \left( x_1 - x_2 \right) + \frac{6R_2 y_2'''}{R_5^2}
\]
From these we find at once that \(U_7 \frac{M_{21}}{M_{22}} = 0\), and we have

\[
(1) \quad \frac{M_{21}}{M_{22}}
\]

as the first solution of \(U_7 = 0\). It is of order 2.

Again, we find that

\[
(2) \quad U_7 \log M_{21} = \frac{R_2 - R_1}{R_3}
\]

\[
(3) \quad U_7 \log M_{31} = (x_2 - x_1) - \frac{6R_6}{M_{31} R_4^2} = \frac{R_1 + R_2}{R_3} - \frac{6R_2}{R_3} \cdot \frac{1}{\sqrt{M_{21} \cdot M_{31}}}
\]

Combining (2) and (3),

\[
(4) \quad U_7 \log (M_{21} \cdot M_{31}) = \frac{2R_2}{R_3} \left(1 - \frac{3}{\sqrt{M_{21} \cdot M_{31}}} \right)
\]

Putting \(\phi = \sqrt{M_{21} \cdot M_{31}}\), (4) becomes

\[
\frac{\phi}{\phi - 3} U_7 \log \phi = \frac{R_2}{R_3}
\]

\[
\therefore \quad \frac{1}{\phi - 3} U_7 (\phi - 3) = \frac{R_2}{R_3}
\]

\[
(5) \quad \therefore \quad U_7 \log (\phi - 3) = \frac{R_2}{R_3}
\]

Similarly, we find that

\[
(6) \quad U_7 \log (\psi - 3) = -\frac{R_1}{R_3}
\]

where \(\psi = \sqrt{M_{22} \cdot M_{32}}\).

Combining (5) and (6), we get

\[
U_7 \log (\phi - 3) (\psi - 3) = \frac{R_2 - R_1}{R_3} = U_7 \log M_{21}
\]

\[
\therefore \quad U_7 \left(\frac{(\phi - 3)(\psi - 3)}{M_{21}}\right) = 0.
\]
Our second solution of $U_7 = 0$ is therefore

$$\left(\sqrt{M_{31} \cdot M_{31} - 3}\right) \left(\sqrt{M_{32} \cdot M_{32} - 3}\right).$$

It is of order 3.

Again, we find

$$U_7 (M_{31} + M_{41}) = (x_2 - x_1)(M_{31} + M_{41}) + \frac{6}{R_3^2} (R_3 y_1''' - R_6)$$

$$= \frac{R_1 + R_2}{R_3} (M_{31} + M_{41}) + \frac{6}{R_4^2} \cdot 3 (x_1 - x_2) R_4^2$$

$$= \frac{R_1 + R_2}{R_3} (M_{31} + M_{41} + 18)$$

$$\therefore U_7 \log (M_{31} + M_{41} + 18) = \frac{R_1 + R_2}{R_3}$$

But from (2)

$$U_7 \log M_{21} = \frac{R_2 - R_1}{R_3}$$

and from (5)

$$U_7 \log (\phi - 3)^2 = \frac{2R_2}{R_3}$$

Combining (8), (9), and (10) we get that

$$U_7 \log \frac{M_{31} (M_{31} + M_{41} + 18)}{(\phi - 3)^2} = 0$$

$$\therefore \frac{M_{31} (M_{31} + M_{41} + 18)}{\left(\sqrt{M_{31} \cdot M_{31} - 3}\right)^2} \text{ is a solution of } U_7 f = 0.$$

Similarly it can be shown that

$$\frac{M_{22} (M_{32} + M_{42} + 18)}{\left(\sqrt{M_{22} \cdot M_{32} - 3}\right)^2}$$

is a solution.

4. Solutions of $U_8 = 0$ to the Fourth Order.

We have now obtained four solutions of $U_7 = 0$, and we proceed to show that these very four are also solutions of $U_8 f = 0$. 
The work follows very similar lines to that for $U_s f = 0$. We find easily that

\[ U_s(R_1) = (y_1 - x_1y_1') R_1 \quad ; \quad U_s(R_2) = (y_2 - x_2y_2' + y_1) R_2 \]

\[ U_s(R_3) = y_1'(y_1 - x_3y_1') - y_2'(y_2 - x_2y_2') \quad ; \]

\[ U_s(R_4) = -3x_1y_1'y_1'' \quad ; \quad U_s(R_5) = -3x_2y_2'y_2'' \quad ; \]

\[ U_s(R_6) = (y_2 - 5x_1y_1') R_6 - 3R_1R_4y_1' \quad ; \quad U_s(R_7) = (y_1 - 5x_2y_2') R_7 - 3R_4R_2y_2' \quad ; \]

\[ U_s(R_8) = -2(y_1 + 4x_2y_1') R_10 + 6y_1''(y_1'y_1''' - 3y_1''') \]

\[ U_s(R_{11}) = -2(y_2 + 4x_2y_2') R_{11} + 6y_2''(y_2'y_2''' - 3y_2''') \]

from which we derive

(13) \[ U_s(M_{21}) = M_{21} \cdot \frac{y_1'R_2 - y_2'R_1}{R_3} \]

(14) \[ U_s(M_{22}) = M_{22} \cdot \frac{y_1'R_2 - y_2'R_1}{R_3} \]

(15) \[ U_s(M_{31}) = M_{31} (y_2 - y_1) - \frac{6R_1y_1'}{R_1^2} \]

(16) \[ U_s(M_{32}) = M_{32} (y_1 - y_2) - \frac{6R_2y_2'}{R_2^2} \]

(17) \[ U_s(M_{41}) = M_{41} (y_2 - y_1) - \frac{6R_1}{R_1^2} (y_1'y_1''' - 3y_1''') \]

(18) \[ U_s(M_{42}) = M_{42} (y_1 - y_2) - \frac{6R_2}{R_2^2} (y_2'y_2''' - 3y_2''') \]

From (13) and (14) we have obviously

(19) \[ U_s \left( \frac{M_{21}}{M_{22}} \right) = 0. \]

From (13) and (15) it follows that

\[ U_s \log (M_{21} \cdot M_{31}) = \frac{2y_1'R_2}{R_2} - 6y_1' \cdot \frac{R_2}{R_3} \cdot \frac{1}{\sqrt{M_{21} \cdot M_{31}}} \cdot \]

\[ \therefore \quad U_s \log \phi = \frac{R_2y_1'}{R_3} \left( 1 - \frac{3}{\phi} \right) \]

(19a) \[ \therefore \quad U_s \log (\phi - 3) = \frac{y_1'R_2}{R_3} \; ; \]
Similarly
\[ U_8 \log (\psi - 3) = -\frac{y_2'R_1}{R_3}; \]
\[ \therefore U_8 \log (\phi - 3) (\psi - 3) = \frac{y_1'R_2 - y_2'R_1}{R_3} = U_8 \log M_{21} \]
\[ \therefore U_8 \log \frac{(\phi - 3) (\psi - 3)}{M_{21}} = 0. \]

Finally, from (15) and (17)
\[ U_8 (M_{31} + M_{41}) = (y_2 - y_1) (M_{31} + M_{41} + 18) \]
hence
\[ (20) \quad U_8 \log (M_{31} + M_{41} + 18) = y_2 - y_1 \]
But
\[ (21) \quad U_8 \log M_{21} = \frac{y_1'R_2 - y_2'R_1}{R_3} \]
and from (19a)
\[ (22) \quad U_8 \log (\phi - 3)^2 = \frac{2R_3y_1'}{R_3} \]
Combining (20), (21), and (22), remembering that \( y_2 - y_1 = \frac{y_2'R_1 + y_1'R_2}{R_3} \),
we get
\[ U_8 \log \frac{M_{21} (M_{31} + M_{41} + 18)}{(\phi - 3)^2} = 0. \]
Thus (11) and its symmetrical counterpart (12) are fourth-order solutions of \( U_8 f = 0 \).

5. Projective Invariants up to the Fourth Order.

We have now obtained four solutions of
\[ U_7 f = 0, \quad U_8 f = 0 \]
which are known to satisfy the six affine equations
\[ U_i f = 0, \quad (i = 1, 2, 3, 4, 5, 6) \]
These four expressions (1), (7), (11), and (12) are therefore invariants of the projective group.
Since the equations extended to the fourth order form a complete system, the number of independent solutions is $12 - 8 = 4$, so that we have them all.

6. PROJECTIVE INVARIANTS OF ORDER 5.

The mixed invariants of orders greater than 4 will involve the affine relative invariants $J_4, J_5, J_6$ for each curve: and we shall distinguish the two curves by adding the subscript 1 or 2. Thus, for example,

$$J_{41} = 3y_1'''y_1^2 - 5y_1''y_1'$$
$$J_{42} = 3y_2'''y_2^2 - 5y_2''y_2'$$

Applying the operators $U_7$ and $U_8$ we find

$$U_7 (J_{41}) = -10x_1J_{41} + 6R_3y_1'''$$
$$U_8 (J_{41}) = -2 (y_1 + 4x_1y_1) J_{41} + 6y_1'' (y_1'y_1'' - 3y_1'')$$

$$U_7 (J_{51}) = -15x_1J_{51} ; U_8 (J_{51}) = -3 (y_1 + 4x_1y_1) J_{51}$$

$$U_7 (J_{61}) = -20x_1J_{61} - 3y_1'''J_{51}$$

$$U_8 (J_{61}) = -4 (y_1 + 4x_1y_1) J_{61} - 3y_1'y_1''J_{51}$$

and the others are obtained from these by changing the subscript 1 to 2, and $R_4$ to $R_5$.

We find from (23) and (25) that

$$U_7 \log \left( \frac{J_{51}}{J_{41}} \right)^{1/3} = -6 \frac{R_3y_1'''}{J_{41}}$$

But we have already

$$U_7 \log M_{41} = (x_2 - x_1) + \frac{6R_3y_1'''}{J_{41}}$$

Thus we have an invariant of order 5, and we can obtain another by changing the subscript 1 to 2.

We find from (23) and (25) that

$$U_7 \log \left( \frac{J_{51}}{J_{41}} \right)^{1/3} = -6 \frac{R_3y_1'''}{J_{41}}$$

But we have already

$$U_7 \log M_{41} = (x_2 - x_1) + \frac{6R_3y_1'''}{J_{41}}$$

whence

$$U_7 \left( \frac{J_{51}}{J_{41}} \frac{M_{41}}{(M_{31} + M_{34} + 18)} \right) = 0 .$$

and the same expression is easily shown to satisfy $U_8 f = 0$.

Thus we have an invariant of order 5, and we can obtain another by changing the subscript 1 to 2.

Before obtaining the last two mixed invariants we prove the following lemma:

"if $I_1$ and $I_2$ are two absolute invariants, then $\frac{dI_1}{dx_1} \div \frac{dI_2}{dx_1}$ is an absolute invariant."

For if $I_1$ is the transform of $I_1$, on differentiating the equation $I_1 = I$ with regard to $X_1$, we get

$$\frac{dI_1}{dX_1} = \frac{dI_1}{dx_1}$$

Similarly

$$\frac{dI_2}{dX_1} = \frac{dI_2}{dx_1}$$

$$\therefore \frac{dI_1}{dX_1} \div \frac{dI_2}{dX_1} = \frac{\frac{dI_1}{dx_1}}{\frac{dI_2}{dx_1}}$$

which proves the lemma.

Now if we multiply the two invariants (11) and (28), invert the product, and extract the square root, we get an invariant

$$I_1 = J_{51}^{-\frac{1}{3}} \cdot \frac{R_3 R_4}{R_2} (\phi - 3)$$

and we have already

$$I_2 = \frac{M_{21}}{M_{22}}$$

$$\frac{dI_1}{dx_1} = -\frac{1}{3} J_{51}^{-\frac{1}{3}} (R_4^{-1} J_6 + \ldots) \cdot \frac{R_3 R_4}{R_2} (\phi - 3) + \ldots \ldots$$

$$\therefore \frac{dI_2}{dx_1} = -\frac{1}{3} \cdot \frac{M_{22}}{M_{21}} \left( \frac{J_{61}}{J_{51}^3} + A \frac{J_{41}^{\frac{1}{3}}}{J_{51}^{\frac{1}{3}}} \right)$$

therefore there is an invariant of the type

$$\frac{J_{61}}{J_{31}^{\frac{1}{3}}} + A \frac{J_{41}^{\frac{1}{3}}}{J_{51}^{\frac{1}{3}}}$$
where \( \lambda \) has still to be determined. The two expressions

\[
\frac{J_{61}^4}{J_{51}^4}, \quad \frac{J_{41}^4}{J_{51}^4}
\]

are affine invariants, therefore \( \lambda \) is an affine invariant.

Now from (25) and (26)

\[
(29) \quad U_7 \left( \frac{J_{61}}{J_{51}^4} \right) = - \frac{3y_{1}''}{J_{51}^4}
\]

\[
(30) \quad U_7 \left( \frac{J_{41}^4}{J_{51}^4} \right) = \frac{3y_{1}''y_{1}'''}{J_{51}^4}
\]

But

\[
U_7 \left( \frac{J_{61}}{J_{51}^4} + \lambda \frac{J_{41}^4}{J_{51}^4} \right) = 0
\]

which in virtue of (29) and (30) is equivalent to

\[
(31) \quad U_7(A) + \frac{3y_{1}''y_{1}'''}{J_{41}} \cdot \lambda = \frac{3y_{1}'''}{J_{41}}
\]

also

\[
U_7 \left( \frac{M_{31}}{M_{41}} \right) = U_7 \left( \frac{B_6}{R_1 J_{41}^4} \right)
\]

\[
= - \frac{3y_{1}'''}{J_{41}^4} - \sqrt{\frac{M_{31}}{M_{41}}} \cdot \frac{3y_{1}''y_{1}'''}{J_{41}}
\]

Comparing (31) and (32) we see that

\[
A = - \sqrt{\frac{M_{31}}{M_{41}}}
\]

\[
\therefore \quad U_7 \left( \frac{J_{61}}{J_{51}^4} - \sqrt{\frac{M_{31}}{M_{41}}} \cdot \frac{J_{41}^4}{J_{51}^4} \right) = 0.
\]

It remains to verify that this expression is also a solution of \( U_8 f = 0 \).

\[
U_8 \left( \frac{M_{31}}{M_{41}} \right) = - \frac{3y_{1}'y_{1}''}{J_{41}^4} - \frac{3y_{1}''y_{1}'''}{J_{41}} \cdot \sqrt{\frac{M_{31}}{M_{41}}}
\]

\[
U_8 \left( \frac{J_{61}}{J_{51}^4} \right) = - \frac{3y_{1}'y_{1}''}{J_{51}^4}
\]

\[
U_8 \left( \frac{J_{41}^4}{J_{51}^4} \right) = - \frac{3y_{1}'y_{1}''}{J_{51}^4}
\]
whence it easily follows that

$$U_8 \left( \frac{J_{61}}{J_{51}} - \sqrt{\frac{M_{31}}{M_{41}}} \cdot \frac{J_{41}^{\frac{1}{3}}}{J_{51}^{\frac{1}{3}}} \right) = 0.$$ 

With a similar expression obtained by interchanging the two curves, we complete our list of invariants.

8. Summary.

To sum up, the mixed invariants of the projective group are as follows:—

(a) One of order two, viz., \( \frac{M_{21}}{M_{22}} \),

(b) One of order three, viz., \( \left( \sqrt{\frac{M_{31} M_{21}}{M_{21}}} - 3 \right) \left( \sqrt{\frac{M_{32} M_{22}}{M_{22}}} - 3 \right) \)

(c) Two of order four, viz.,

\[
\frac{M_{21} (M_{31} + M_{41} + 18)}{(\sqrt{M_{21} M_{21}} - 3)^2} \quad \frac{M_{22} (M_{32} + M_{42} + 18)}{(\sqrt{M_{22} M_{32}} - 3)^2}
\]

(d) Two of order five, viz.,

\[
\frac{J_{51} \cdot M_{41}}{J_{41} (M_{51} + M_{41} + 18)} \quad \frac{J_{52} \cdot M_{42}}{J_{42} (M_{52} + M_{42} + 18)}
\]

(e) Two of order six, viz.,

\[
\frac{J_{61}}{J_{51}} - \sqrt{\frac{M_{31}}{M_{41}}} \cdot \frac{J_{41}^{\frac{1}{3}}}{J_{51}^{\frac{1}{3}}} \quad \frac{J_{62}}{J_{52}} - \sqrt{\frac{M_{32}}{M_{42}}} \cdot \frac{J_{42}^{\frac{1}{3}}}{J_{52}^{\frac{1}{3}}}.
\]

No matter how varied the configuration of analytic arcs, all the projective invariants are functions of these types, and of the types found by Halphen for a single analytic arc.

The geometrical meaning of the mixed invariants will be considered in another paper.

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