AFFINE DIFFERENTIAL INVARIANTS OF TWO ANALYTIC ARCS.

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INTRODUCTION.

Some years ago the writer showed* that for any $r$-parameter-group in the plane the most general differential configuration of analytic arcs from the point of view of invariants consists of two arcs at distinct points; that is to say, that no new independent types are introduced by taking a more complicated configuration.

The invariants of this fundamental configuration are of two kinds:—

(a) those involving elements from one arc only,
(b) those involving elements from both arcs.

It is well known that there are an infinite number of type (a) for each arc; one of order $r$, one of order less than $r$, and the rest derived from these by successive differentiations. In the above paper the writer also proved that the number of invariants of type (b) for an $r$-parameter group is precisely $r$. We shall call this type mixed.

The object of the present paper is to determine the mixed invariants for the affine group of transformations in the plane, and to interpret them geometrically.

This represents the first step towards the solution of the wider and more interesting problem, viz., the finding of all the projective invariants of a differential configuration.

1. INFINITESIMAL TRANSFORMATIONS OF THE GROUP.

We take as our fundamental configuration two points $P (x_1, y_1)$ and $Q (x_2, y_2)$ through each of which passes an analytic arc. Derivatives at these points will be distinguished by the subscripts 1 and 2. Thus $y_{1'''}$ will represent the third derivative of $y_1$ with regard to $x_1$.

The Affine Group for this fundamental configuration consists of the four equations—

\[
\begin{align*}
X_i &= ax_i + by_i + c \\ Y_i &= dx_i + ey_i + f
\end{align*}
\]  

($i = 1, 2$)

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The two invariants of the first type for each are known to be of orders 5 and 6 respectively. To find the mixed invariants it will therefore only be necessary to extend the infinitesimal transformations to the fourth order. We find them to be

\[ V_1f = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2}; \quad V_2f = \frac{\partial f}{\partial y_1} + \frac{\partial f}{\partial y_2} \]

\[ V_3f = \sum x \frac{\partial f}{\partial x} - \sum y \frac{\partial f}{\partial y} - 2 \sum y'' \frac{\partial f}{\partial y''} - 3 \sum y''' \frac{\partial f}{\partial y'''} - 4 \sum y'''' \frac{\partial f}{\partial y''''} \]

\[ V_4f = \sum y \frac{\partial f}{\partial x} - \sum y'' \frac{\partial f}{\partial y''} - 3 \sum y''' \frac{\partial f}{\partial y'''}, \quad \sum (4y'''' + 3y''''') \frac{\partial f}{\partial y'''''} \]

\[ V_5f = \sum x \frac{\partial f}{\partial y} + \sum \frac{\partial f}{\partial y''} \]

\[ V_6f = \sum y \frac{\partial f}{\partial y} + \sum y' \frac{\partial f}{\partial y'} + \sum y'' \frac{\partial f}{\partial y''} + \sum y''' \frac{\partial f}{\partial y'''} + \sum y'''' \frac{\partial f}{\partial y''''} \]

The absolute invariants are the solutions of

\[ V_1f = V_2f = V_3f = V_4f = V_5f = V_6f = 0. \]

These equations form a complete system; the number of variables is 12 and the number of equations 6, therefore the number of independent solutions is \(12 - 6 = 6\). These will necessarily be of the mixed type.

2. Solution of the Equations.

We shall first consider a convenient sub-group

\[ X_i = ax_i + c \]

\[ Y_i = dy_i + f \quad (i = 1, 2) \]

whose infinitesimal transformations extended to the fourth order are \( V_1f, V_2f, V_3f, V_4f \). The equations

\[ V_1f = V_2f = V_3f = V_4f = 0 \]

form themselves a complete system with 8 independent solutions, of which 4 are of the mixed type. The four solutions of type (a) may be taken to be

\[ \frac{y_i'''}{y_i'}, \quad \frac{y_i''''}{y_i'''} \quad (i = 1, 2) \]
which are obviously independent. The fact that these are of orders 3 and 4 respectively shows that the mixed invariants of (2) can be obtained by considering only terms of the equations (1) not exceeding the second order.

Solving \( V_{af} = 0 \), we get an independent set of solutions.

\[
x_1, x_2, \theta_1 = \frac{y_1}{y_2}, \quad \theta_2 = \frac{y_1}{y_1'}, \quad \theta_3 = \frac{y_1}{y_2'}, \quad \theta_4 = \frac{y_1'}{y_1''}, \quad \theta_5 = \frac{y_1'}{y_2''}
\]

Taking these for new variables, our equations (3) reduce to the three

\[
\frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2} = 0
\]

\[
(\theta_1 - \theta_1^2) \frac{\partial f}{\partial \theta_1} + \theta_3 \frac{\partial f}{\partial \theta_2} + \theta_3 \frac{\partial f}{\partial \theta_3} + \theta_4 \frac{\partial f}{\partial \theta_4} + \theta_5 \frac{\partial f}{\partial \theta_5} = 0
\]

\[
x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \theta_2 \frac{\partial f}{\partial \theta_2} + \theta_3 \frac{\partial f}{\partial \theta_3} + 2\theta_4 \frac{\partial f}{\partial \theta_4} + 2\theta_5 \frac{\partial f}{\partial \theta_5} = 0
\]

Again, a complete set of solutions of the first of these three is

\[
\psi = x_1 - x_2, \quad \theta_1, \theta_2, \theta_3, \theta_4, \theta_5
\]

and with these as new variables the third becomes

\[
\psi \frac{\partial f}{\partial \psi} + \theta_2 \frac{\partial f}{\partial \theta_2} + \theta_3 \frac{\partial f}{\partial \theta_3} + 2\theta_4 \frac{\partial f}{\partial \theta_4} + 2\theta_5 \frac{\partial f}{\partial \theta_5} = 0
\]

while the second remains unchanged.

Finally, putting

\[
X_1 = \frac{\psi}{\theta_2}, \quad X_2 = \frac{\psi}{\theta_3}, \quad X_3 = \frac{\psi}{\sqrt{\theta_4}}, \quad X_4 = \frac{\psi}{\sqrt{\theta_5}}
\]

we arrive at the single equation

\[
(\theta_1 - \theta_1^2) \frac{\partial f}{\partial \theta_1} - X_1 \frac{\partial f}{\partial X_1} - X_2 \frac{\partial f}{\partial X_2} - \frac{1}{2} X_3 \frac{\partial f}{\partial X_3} - \frac{1}{2} X_4 \frac{\partial f}{\partial X_4} = 0,
\]

which has a set of independent solutions

\[
X_1, \quad X_1, \quad X_1, \quad \frac{1 - \theta_1}{\theta_1 X_1}
\]
or, in terms of the original variables

\[
\frac{y_1'}{y_2'} = \frac{1}{x_1 - x_2} \frac{y_1'}{y_1''} = \frac{1}{x_1 - x_2} \frac{y_1'}{y_2''} = \frac{y_1 - y_2}{(x_1 - x_2) y_1'}
\]

The eight expressions (4) and (5), which we shall write

\[
\omega_1, \omega_2, \omega_3, \omega_4; \omega_5, \omega_6, \omega_7, \omega_8
\]

constitute the complete solution of the system

\[
V_1 f = V_2 f = V_3 f = V_4 f = 0
\]

We have now to find what six functions of these quantities satisfy the two remaining equations \(V_5 f = V_3 f = 0\). Taking the \(\omega\)'s as new variables, \(V_5 f = 0, V_3 f = 0\) become respectively

\[
\omega_6 (3\omega_1^2 - \omega_1) \frac{\partial f}{\partial \omega_1} + (3\omega_2^2 - \omega_2) \frac{\partial f}{\partial \omega_2} + \omega_5 \left( -2\omega_3 + 10 \frac{\omega_5}{\omega_1} \right) \frac{\partial f}{\partial \omega_3}
+ \left( -2\omega_4 + 10 \frac{\omega_4^2}{\omega_2} \right) \frac{\partial f}{\partial \omega_4} + \omega_5 \left( 1 - \omega_5 \right) \frac{\partial f}{\partial \omega_5} + \omega_6 (2\omega_6 + \omega_6 \omega_8) \frac{\partial f}{\partial \omega_6}
+ \omega_5 \left( -\omega_2 \omega_8 - \omega_7 + 3 \frac{\omega_7}{\omega_5} \right) \frac{\partial f}{\partial \omega_7} + \omega_5 (\omega_8 - \omega_8^2) \frac{\partial f}{\partial \omega_8} = 0,
\]

and

\[
-\omega_1 \frac{\partial f}{\partial \omega_1} - \omega_2 \omega_3 \frac{\partial f}{\partial \omega_2} - 2\omega_3 \frac{\partial f}{\partial \omega_3} - 2\omega_4 \omega_5 \frac{\partial f}{\partial \omega_4} + \omega_5 (1 - \omega_5) \frac{\partial f}{\partial \omega_5}
+ \omega_6 \frac{\partial f}{\partial \omega_6} + \omega_7 \frac{\partial f}{\partial \omega_7} + (1 - \omega_8) \frac{\partial f}{\partial \omega_8} = 0.
\]

Solving the last, we get

\[
\lambda_1 = \frac{\omega_1^2}{\omega_3}, \quad \lambda_2 = \omega_1 \omega_6, \quad \lambda_3 = \omega_1 \omega_7, \quad \lambda_4 = 1 - \omega_8
\]

\[
\lambda_5 = \frac{1 - \omega_5}{\omega_1 \omega_5}, \quad \lambda_6 = \frac{\omega_2^2}{\omega_4}, \quad \lambda_7 = \frac{1 - \omega_5}{\omega_2}
\]

and substituting in the other, the system finally reduces to the single equation

\[
(6\lambda_1 - 10) \frac{\partial f}{\partial \lambda_1} + \lambda_2 (3 + \lambda_4) \frac{\partial f}{\partial \lambda_2} + \lambda_3 (3 + \lambda_4 + 3\lambda_5) \frac{\partial f}{\partial \lambda_3}
+ \lambda_4 (\lambda_4 - 3) \frac{\partial f}{\partial \lambda_4} - \lambda_5 (\lambda_5 + 3) \frac{\partial f}{\partial \lambda_5} + \lambda_5 (6\lambda_6 - 10) \frac{\partial f}{\partial \lambda_6} + \lambda_5 (\lambda_7 - 3) \frac{\partial f}{\partial \lambda_7} = 0
\]
the solutions of which are

\[
\frac{(\lambda_4 - 3)^2}{\lambda_4^2 (6\lambda_4 - 10)} = \frac{\{R_1 y_1''' + 3 (x_1 - x_2) y_1''/y_1^{\prime/3} R_1\}^2}{y_1^{\prime/3} R_1}
\]

\[
\frac{(\lambda_7 - 3)^2}{\lambda_7^2 (6\lambda_7 - 10)} = \frac{\{(y_1' - y_2') y_2''' + 3 y_2''/y_1^{\prime/2} (3 y_2''/y_1^2 - 5 y_2''/y_1^2)\}^2}{(y_1' - y_2')^2 (3 y_2''/y_1^2 - 5 y_2''/y_1^2)}
\]

\[
\frac{(\lambda_4 - 3)}{\lambda_4} \frac{\lambda_5}{\lambda_5 + 3} = \frac{R_2 y_1''' + 3 (x_1 - x_2) y_1''/y_1^{\prime/2} R_1}{(y_1' - y_2') y_1''' - 3 y_1''/y_1^{\prime/2}}
\]

\[
\frac{(\lambda_4 + 3)}{\lambda_4} \frac{(\lambda_4 - 3)}{\lambda_4} \frac{\lambda_5^3 \lambda_3}{\lambda_2} = \frac{(R_3 y_1''' + 3 (x_1 - x_2) y_1''/y_1^{\prime/5} y_2^{\prime/2})^3}{y_1^{\prime/2} y_2^{\prime/2}}
\]

where

\[R_1 = (y_1 - y_2) - (x_1 - x_2) y_1'.\]

These are the mixed absolute invariants of the affine group. It will be noticed that they are homogeneous and isobaric expressions of "degree" and of "weight" zero, if we define the "weight" of \(y^{(n)}\) to be \(n + 1\), and that of \(x_1 - x_2\) to be \(-1\), while the "degree" of \(x_1 - x_2\) is taken to be zero.

They might have been built up, using these properties, from the following relative invariants;

\[R_1 = y_1 - y_2 - (x_1 - x_2) y_1', \quad R_2 = y_2 - y_1 - (x_2 - x_1) y_2', \quad R_3 = y_1' - y_2', \quad R_4 = y_1'', \quad R_5 = y_2'' \]

\[R_6 = y_1'' + 3 (x_1 - x_2) y_1''/y_1', \quad R_7 = y_2'' + 3 (x_2 - x_1) y_2''/y_2' \]

\[R_8 = (y_1' - y_2') y_1''' - 3 y_1''/y_1', \quad R_9 = (y_2' - y_1') y_2''' - 3 y_2''/y_2' \]

\[R_{10} = 3 y_1''' y_1^4 - 5 y_1''^2, \quad R_{11} = 3 y_2''' y_2^4 - 5 y_2''^2 \]

The vanishing of each of these relative invariants indicates a specialisation of the fundamental configuration which remains invariant under the affine group. If we note that the slope of the axis of the osculating parabola is

$$\frac{y'y''' - 3y''^2}{y'''}$$

we may interpret the equations as follows:—

- \( R_1 = 0 \) or \( R_2 = 0 \). The line joining the given points \((x_1, y_1)\) and \((x_2, y_2)\) is tangent to one of the given arcs.
- \( R_3 = 0 \). The tangents to the arcs are parallel.
- \( R_4 = 0 \) or \( R_5 = 0 \). One of the tangents has three-point contact.
- \( R_6 = 0 \) or \( R_7 = 0 \). The line joining the points is parallel to the axis of the osculating parabola of one of the arcs.
- \( R_8 = 0 \) or \( R_9 = 0 \). The tangent to one of the curves is parallel to the axis of the osculating parabola of the other.
- \( R_{10} = 0 \) or \( R_{11} = 0 \). The osculating parabola of one of the curves hyper-osculates that curve.

The Absolute Invariants.

The six absolute invariants may be combined so as to give symmetrical pairs. Two will be of order 2, two of order 3, and two of order 4. These are

- \( M_{21} = \frac{R_2 R_4}{R_1 R_3^2} \), \( M_{32} = \frac{R_1^2 R_5}{R_2 R_3^2} \);
- \( M_{31} = \frac{R_6^2}{R_1 R_4^3} \), \( M_{32} = \frac{R_7^2}{R_2 R_5^3} \);
- \( M_{41} = \frac{R_1 R_{10}}{R_4^3} \), \( M_{42} = \frac{R_2 R_{11}}{R_5^3} \).

In this form their independence is clearly exhibited.

The invariance of these functions of the general configuration of two arbitrary analytic arcs corresponds to the invariance of certain geometric entities, as follows:—
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Let $P$ be the point $(x_1, y_1)$; $Q$ the point $(x_2, y_2)$; $T$ the point of intersection of the tangents at $P$ and $Q$; $R$ the foot of the perpendicular from the centre of curvature of the arc at $P$ on $PQ$. We easily find that

$$-M_{21} = \frac{PT^2}{PQ \cdot PR}.$$  

This geometric quantity is therefore invariant under affine transformations. The interpretation is not altogether satisfying from the affine point of view, since the circle of curvature is used.

Let us consider the family of parabolas having three-point contact with the curve at $P$. Their equation is

$$(6) \quad y_1'' \{a (x - x_1) - (y - y_1)\}^2 + 2y_1' (a - y_1')^2 (x - x_1) - 2 (a - y_1')^2 (y - y_1) = 0.$$  

where $a$ is a parameter. The condition that $Q$ lies on one of these parabolas is that $a$ satisfies the condition

$$(7) \quad (a - y_1')^2 = 0.$$  

Comparing this equation with $M_{21} + 2 = 0$, that is,

$$y_1'' \{(y_2 - y_1) - (x_2 - x_1) y_2'\}^2 + 2 (y_1' - y_2')^2 R_1 = 0,$$

we see that they are identical if $a = y_1'$. The equation $M_{21} + 2 = 0$ therefore means that the tangent at $Q$ is parallel to the axis of either of the two parabolas which pass through $Q$ and have three-point contact with the arc at $P$. This is therefore an affine invariant property.

We may look at the matter a little differently. Let the two values of $a$ given by (7) be denoted by $y_3'$ and $y_4'$ respectively. These are the slopes of the axes of the two parabolas of the family (6) which pass through $Q$. If we denote by $\lambda$ either the cross-ratio $[y_1' y_2' y_3' y_4']$ or its reciprocal we find that

$$M_{21} = -2 \left( \frac{1 + \lambda}{1 - \lambda} \right)^2,$$

from which the above property may be verified by putting $\lambda = 0$ or $\infty$. Moreover, if $PQ$ is tangent to the second arc at $Q$, the above cross-ratio is harmonic, and $M_{21} = 0$. 
Through $Q$ draw a line parallel to the axis of the osculating parabola at $P$ meeting $PT$ in $E$. Then

$$M_{31} = \frac{9PE^2}{PQ \cdot PR}$$

Again the use of the circle of curvature gives a somewhat unnatural interpretation.

The osculating parabola at $P$ has for equation

\[(8) \quad [(y_1'y_1''' - 3y_1''') (x - x_1) - y_1''' (y - y_1)]^2 - 18y_1''' [(y - y_1) - (x - x_1)] y_1' = 0.\]

The condition that this passes through $Q (x_2, y_2)$ is clearly \( R_6^2 + 18 R_1 R_4^3 = 0, \) that is

$$M_{31} + 18 = 0,$$

which is therefore the equation of the osculating parabola at $P$ with $Q$ as current point.

Through $P$ draw $PV$ parallel to the axis of the osculating parabola (8), and through $Q$ draw $QV$ parallel to the tangent at $P$. Then if $S$ be the focus, we find

$$QV^2 = \frac{(1 + y_1''') R_6^2}{9R_4^3}$$

$$SP = \frac{3}{2} \cdot \frac{(1 + y_1''') R_4}{\sqrt{y_1''''^2 + (y_1'y_1''' - 3y_1''')^2}}$$

$$PV = -\frac{1}{3} R_1 \frac{\sqrt{y_1''''^2 + (y_1'y_1''' - 3y_1''')^2}}{R_4^3}$$

whence

$$\frac{QV^2}{4SP \cdot PV} = -\frac{1}{18} \frac{R_6^2}{R_1 R_4^3} = -\frac{1}{18} M_{31}$$

Again it is clear that \( M_{31} + 18 = 0 \) gives the equation of the osculating parabola at $P$.

It follows without difficulty that the equation \( M_{31} = \text{const.} \), that is

$$\frac{QV^2}{4SP \cdot PV} = \text{const.}$$

regarded as the locus of $Q$ is a family of parabolas tangent to the osculating parabola at $P$, and having their axes parallel to its axis, and their foci on $SP$.

The use of the focus is, however, open to the same objection as that of the centre of curvature.
Join $Q$ to the centre $O$ of the osculating five-pointic conic (ellipse or hyperbola) at $P$. Let the tangent at $P$ meet $OQ$ in $F$. We find easily that

$$M_{41} = \frac{9FQ}{OF}.$$

It is not difficult to see that this ratio is invariant under affine transformation, for the three points $O, F, Q$ transform into the corresponding points.

The equation $M_{41} = \text{const.}$, regarded as the locus of $Q$, represents a family of straight lines parallel to the tangent at $P$ ($M_{41} = 0$). The other line of the family tangent to the osculating conic is

$$M_{41} + 18 = 0.$$