Macroscopic Greenberger-Horne-Zeilinger and W states in flux qubits

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We investigate two types of genuine three-qubit entanglement, known as the Greenberger-Horne-Zeilinger (GHZ) and W states, in a macroscopic quantum system. Superconducting flux qubits are theoretically considered in order to generate such states. A phase coupling is proposed to offer enough strength of interactions between qubits. While an excited state can be the W state, the GHZ state is formed at the ground state of the three flux qubits. The GHZ and W states are shown to be robust against external flux fluctuations for feasible experimental realizations.

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Entanglement plays a crucial role in quantum information science. Controllable quantum systems such as photons, atoms, and ions have provided the opportunities to generate the entanglements. Recent experiments on two qubits have shown the existence of entanglement in different types of microscopic systems. Further, multipartite entanglements such as the Greenberger-Horne-Zeilinger† (GHZ) and W (Refs. 2 and 3) states have been demonstrated in recent experiments of atoms,4 photons, and trapped ions.5,6 However, in solid-state qubits, it has not yet been achieved.

As a macroscopic quantum system, superconducting qubit systems have been investigated intensively in experiments because their system parameters can be controlled to manipulate quantum states coherently. Indeed, the entanglements between two charges,7 phase,8,9 and flux qubits10,11 have been reported. While the timely evolving states in the experiments of charge qubits7 exhibit a partial entanglement, the excited level (eigenstate) of capacitively coupled two phase qubits9 shows higher fidelity for the entanglement. The experiments in Ref. 10 show a possibility that two flux qubits can be entangled by a macroscopic quantum tunneling between two-qubit states, flipping both qubits. Actually, the higher fidelity in the capacitively coupled two phase qubits is caused by the two-qubit tunneling processes.9 In a very recent study, the two-qubit tunneling process was theoretically shown to play an important role in generating the Bell states, maximally entangled, in the ground and excited states.12

For multipartite entanglements in superconducting qubit systems, there have been few studies. To produce the GHZ state in three charge qubits, only a way of doing a local qubit operation via time evolutions was suggested.13 As one of the possible directions to produce such multipartite entanglements, then it is natural to ask how to create the W state as well as the GHZ state in the eigenstates of superconducting three-qubit systems. Here, we consider three flux qubits. Normally, the interaction strength between inductively coupled flux qubits14 is not so strong that the controllable range of interaction is not sufficiently wide. To control a wide range of interaction strengths in the qubits, we use the phase-coupling scheme15–19 for three qubit [see Fig. 1(a)] which enables one to generate the GHZ and W states and to keep them robust against external flux fluctuations for feasible experimental realizations.

We start with the model shown in Fig. 1(a). The Hamiltonian is written by $H = \frac{1}{2} \hat{p}_i^2 M_{ij}^{-1} \hat{p}_j + U_{\text{eff}}(\phi)$, where $\hat{p}_j = -i\hbar \partial / \partial \phi_j$ and $M_{ij} = (\Phi_0 / 2 \pi)^2 C_i \delta_{ij}$ with the capacitance of the Josephson junctions $C_i$. The dynamics of the flux qubits20 are described by the phase variables $\phi_j = (\phi_a, \phi_b, \phi_c)$ with $q = a, b, c$ and $i = 1, 2, 3$, where $\phi$’s are the phase differences across the Josephson junctions. If we neglect the small inductive energy, the effective potential is written in terms of the Josephson junction energies, $U_{\text{eff}}(\phi) = \sum_{j=1}^{3} \sum_{q=r,s} E_j (1 - \cos \phi_q) + E_j (1 - \cos \phi'_q)$. The periodic boundary conditions involved in the qubit loops and the connecting loops can be written as

$$\phi_{q1} + \phi_{q2} + \phi_{q3} = 2 \pi (n_q + f_q),$$

$$\phi_{a1} - \phi_{a2} - \phi_{b1} + \phi_{c1} = 2 \pi r,$$

$$\phi_{b1} - \phi_{b2} - \phi_{c1} + \phi_{c2} = 2 \pi s,$$

where $q = a, b, c$ is qubit index and $r, s$, and $n_q$ integers. Here, $f_q = \Phi_0 / \Phi_0$ with external flux $\Phi_0$ and the superconducting unit flux quantum $\Phi_0 = h / 2e$. Two independent conditions in Eqs. (2) and (3) are the boundary conditions for connecting loops. For simplicity, we consider $E_{j1} = E_{j2} = E_j$ and $C_1 = C_2 = C_3$, so we can set $\phi_{q2} = \phi_{q3}$ and Eq. (1) becomes $\phi_{j1} = 2 \pi (n_q + f_q) - 2 \phi_{q1}$. The results for $E_{j1} \neq E_{j2}$ are qualitatively the same.

At the coalescence point $(f_{a1}, f_{b1}, f_{c1}) = (1/2, 1/2, 1/2)$, the effective potential is given by

$$U_{\text{eff}}(\phi) = \sum_{q=a,b,c} \left( E_{j1} \cos \phi_{q1} + E_{j2} \cos \phi_{q3} - E_j \cos \phi'_{q1} \right) + 3 E_{j1} + 6 E_j + 3 E_j'. $$

Here, we introduce a rotated coordinates $\tilde{\phi} = (\phi_a, \phi_b, \phi_c)$ in Fig. 1(b). The Euler rotations provide new coordinates such as $\tilde{\phi} = R_{\chi}(\chi, 0, 0) R_{\theta}(0, 0, \theta) \phi = R(\chi, \theta) \phi^T$, with $\chi = -\tan^{-1} \sqrt{2}$, $\theta = -\pi / 4$, and $\phi = (\phi_{a3}, \phi_{b3}, \phi_{c3})$, which can be written explicitly as
\[
\begin{pmatrix}
\phi_a \\
\phi_b \\
\phi_c
\end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix}
\sqrt{3} & -\sqrt{3} & 0 \\
1 & 1 & -2 \\
\sqrt{2} & \sqrt{2} & \sqrt{2}
\end{pmatrix} \begin{pmatrix}
\phi_{a3} \\
\phi_{b3} \\
\phi_{c3}
\end{pmatrix}.
\]

In the same way, a new coordinates for \( \varphi' = (\phi'_a, \phi'_b, \phi'_c) \) is given as \( \varphi' = R(\chi, \theta) \varphi \), with \( \varphi = (\phi'_a, \phi'_b, \phi'_c) \). Using the boundary conditions of Eqs. (2) and (3), the Hamiltonian is written in the transformed coordinates, \( \hat{H} = (\phi_a, \phi_b, \phi_c, \phi'_a, \phi'_b, \phi'_c) \), as

\[
\frac{\partial \hat{H}}{\partial \theta} = \sum_{\mu=0,1,2} M_{\mu} + \frac{\partial \hat{P}}{\partial \phi'}, \quad \text{where} \quad M_{\mu} = M_{\beta} = 4C_1 + 2C' + 2C, \quad M_{\gamma} = 4C_1 + 2C, \quad \text{and} \quad M_{\mu} = C'.
\]

Note that the value of \( \phi' \) is determined at the potential minimum, \( \partial U(\phi')/\partial \phi' = 0 \).

The eight corners of the hexahedron in Fig. 1(b) correspond to the three-qubit states. Here, the \([|0\rangle \langle 0|\) is defined as diamagnetic (paramagnetic) current state which corresponds to positive (negative) value of \( \phi_a \) in the boundary condition of Eq. (1). These states can be represented more clearly in the rotated coordinates \( (\phi_a, \phi_b, \phi_c) \) by the effective potential \( U(\phi') \). The threefold rotational symmetry of the \( \phi_a \) axis, which can be shown as in the following. Using the transformation of Eq. (5), one of the terms in Eq. (4) is written as \( \sum_{\mu=0,1,2} \cos \phi_{a3} = \cos(\phi'_b/\sqrt{3})[2 \cos(\phi'_b/\sqrt{6}) + \cos(2\phi'_b/\sqrt{6})] \).

In the same way, a new coordinates for \( \varphi' = (\phi'_a, \phi'_b, \phi'_c) \) is given as \( \varphi' = R(\chi, \theta) \varphi \), with \( \varphi = (\phi'_a, \phi'_b, \phi'_c) \). Using the transformation of Eq. (5), one of the terms in Eq. (4) is written as \( \sum_{\mu=0,1,2} \cos \phi_{a3} = \cos(\phi'_b/\sqrt{3})[2 \cos(\phi'_b/\sqrt{6}) + \cos(2\phi'_b/\sqrt{6})] \).

Here, if we rotate the potential by \( 2\pi/3 \) about the \( \phi_b \) axis as \( \psi_b = -1/2 \phi_b - \sqrt{3}/2 \phi_c \), \( \psi_b = (1/2 \phi_a - 1/2 \phi_b) \), and \( \psi_c = \phi_c \), we can easily check the invariance of the effective potential, \( U(\phi') \).

In order to study the GHZ state, \( |\Psi_{\text{GHZ}}\rangle = (|000\rangle + |111\rangle)/\sqrt{2} \), we draw the yellow (dark gray) square introducing the auxiliary coordinates defined by \( \phi_b = (\phi_{a3} + \phi_{b3})/\sqrt{2} \), and \( \phi_{a3} = (\phi_{a3} - \phi_{b3})/\sqrt{2} \), where for state \( |W\rangle = (|001\rangle + |110\rangle + |101\rangle + |010\rangle)/\sqrt{3} \), we consider the blue (light gray) triangle. Figure 1(c)–1(e) show the effective potential \( U(\phi') \) in Eq. (4). The three qubits are decoupled for \( E'_j = 0 \). Fig. 1(c) shows that the single-qubit tunneling \( t_{ij} \) is dominant over the three-qubit tunneling \( t'_{ij} \). As \( E'_j \) increases, it is shown in Fig. 1(d) that the three-qubit tunneling becomes dominant. Then, the GHZ state is expected to be formed at the ground state. The dotted line in Fig. 1(d) coincides with \( \phi_c \) axis in Fig. 1(b). Along the \( \phi_c \) axis, the double-well potential is given by \( U_{\text{eff}}(0,0,0,0,0,0) = 3E_{ij}(1 + \cos(\phi_c/2) + 6E_{ij}(1 - \cos(\phi_c/2) \), where the barrier height is proportional to \( E_{ij} \). The WKB approximation allows us to calculate the three-qubit tunneling \( t_{ij} \) through this double-well potential.21,22 Other tunnelings such as single-qubit tunnelings, \( t_{ij} \) and \( t_{ij} \), and two-qubit tunnelings, \( t_{ij} \) and \( t_{ij} \), can also be calculated. The tight-binding approximation based on the eight states of three qubits gives the effective Hamiltonian, \( H = \sum \langle \psi_i | \hat{H} | \psi_i \rangle |\psi_i\rangle \langle \psi_i | - \sum \langle t_{ij} | \hat{H} | t_{ij} \rangle |\psi_i\rangle \langle \psi_i | - \sum \langle t_{ij}^* | \hat{H} | t_{ij}^* \rangle |\psi_i\rangle \langle \psi_i | \rangle \), with \( t_{ij} = t_{i\mu}^{(b)} \) and \( |\psi_i\rangle = |s_{ij,\mu,\rho}\rangle \), with \( s_{ij,\mu,\rho} \in \{0,1\} \).

The global entanglement for tripartite systems can be quantified by the \( Q \) measure.23 For a normalized arbitrary three-qubit state, \( |\Psi\rangle = c_{000} |000\rangle + c_{001} |001\rangle + c_{010} |010\rangle + c_{011} |011\rangle + c_{100} |100\rangle + c_{101} |101\rangle + c_{110} |110\rangle + c_{111} |111\rangle \), the \( Q \) factor is given by \( Q(|\Psi\rangle) = 4/3 \sum D_i(|\Psi\rangle) \), where \( D_i(|\Psi\rangle) \)
FIG. 2. (Color online) The $Q$ factors of the ground state in the three qubit system for (a) $E'_1/E_j=0.02$ and for (b) $E'_1/E_j=0.05$. Here, $f_0=\sqrt{3}/2$ and $E_j/E_0=0.7$. (c) Cut view of $Q$ factors in (a) and (b) for $f_0=0$. (d) For $f_0=f_B=0$ and $E_j/E_0=0.6$, $Q$ factors are plotted as a function of $f_j$, for several $E_j$.

TABLE I. Peak widths for $Q$ factors of GHZ state in Fig. 2(d) and of $W$ state in Fig. 3(d) at 95% of the maximum values. Here, the unit of $E'_j$, $E_j$, and $t$ is $E_j$.

<table>
<thead>
<tr>
<th>$E'_j$</th>
<th>$E_j$</th>
<th>$f'_j$</th>
<th>Peak width</th>
<th>$f_B$</th>
<th>Peak width</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\delta f'_{j}$</td>
<td>$\delta f_B$</td>
<td>$\delta W$</td>
</tr>
<tr>
<td>0.05</td>
<td>0.7</td>
<td>7.0 $\times$ 10^{-6}</td>
<td>$\sim$ 5 $\times$ 10^{-7}</td>
<td>6.3 $\times$ 10^{-4}</td>
<td>$\sim$ 10^{-4}</td>
</tr>
<tr>
<td>0.1</td>
<td>0.75</td>
<td>2.6 $\times$ 10^{-7}</td>
<td>$\sim$ 2 $\times$ 10^{-8}</td>
<td>1.0 $\times$ 10^{-4}</td>
<td>$\sim$ 2 $\times$ 10^{-5}</td>
</tr>
<tr>
<td>0.6</td>
<td>0.58</td>
<td>5.1 $\times$ 10^{-3}</td>
<td>$\sim$ 4 $\times$ 10^{-4}</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

ever, for larger value of $E'_j/E_j=0.6$, we obtained $f'_j/f_B^2=6.4$ with $f'_j/E_j=8.0 \times 10^{-4}$. Hence, for the strong coupling case, we can expect higher $Q$ factor for the GHZ state.

In Table I, the peak widths for both GHZ and $W$ states are calculated at 95% of the maximum value of $Q$ factor, which are approximately proportional to $f'_j$ and $f_B^2$, respectively. During the Rabi oscillations, the fluctuation of flux is estimated to be in the order of $10^{-6}$ for $H/E_1^2$ (Ref. 24) and $1/f$ critical current fluctuations of the Josephson junctions is rather weak. In recent experiments for flux qubits, the flux amplitudes are controlled up to the accuracy of $10^{-8}$. In this respect, the peak width, $\delta f'_j \sim 4 \times 10^{-4}$, for $E_j/E_0=0.6$ will be sufficient to observe the GHZ state experimentally.

In Fig. 1(b), we present the blue (light gray) triangle whose corners correspond to the three states consisting of the $W$ state, $|\Psi_W\rangle=(|\uparrow \downarrow \downarrow\rangle+|\downarrow \uparrow \uparrow\rangle+|\downarrow \downarrow \uparrow\rangle)/\sqrt{3}$. The blue (light gray) triangle intersects $f_0$ axis at $f_0>0$. Actually, there is another intersection plane with $f_0<0$ for another possible $W$ state. For simplicity, we will focus on the $W$ state on the blue (light gray) triangle plane. The effective potential at the plane of the blue (light gray) triangle for the three states is shown in Fig. 1(e). Energetically, in our model, the energies of three states are higher than those of the two states consisting of the $GHZ$ state, i.e., the ground state. Then, the $W$ state can be observed in an excited state.

Let us discuss how a $W$ state can be realized in an excited state. At the coarsenness point $(f_0,f_B,f_3)=(0,0,\sqrt{3}/2)$, the six states except for $\{|\downarrow\downarrow\downarrow\rangle, |\uparrow\uparrow\uparrow\rangle\}$ are degenerated in the second excited state. The six states are classified into two classes, $\{|\uparrow\downarrow\downarrow\rangle, |\downarrow\uparrow\uparrow\rangle\}$ and $\{|\uparrow\uparrow\downarrow\rangle, |\downarrow\downarrow\uparrow\rangle\}$ with $S_z=1/2$ and $\{|\uparrow\uparrow\uparrow\rangle, |\downarrow\downarrow\downarrow\rangle\}$ with $S_z=-1/2$. Hence, the two classes of the six states can be separated by applying an additional flux $\Delta f_3$. The three states of each class can form a $W$ state. As shown in Fig. 1(e), the two-qubit tunneling amplitude $f'_3$ creates the $W$ state, while the single-qubit tunneling $f'_2$ destroys the $W$ state because it induces a superposition of states of the two classes. If other small tunnelings are negligible, then the $Q$ factor is given by $Q(|\Psi\rangle)=8/[1+2.5(f'_2/\Delta E^2)+9 \times (1+4(f'_2/\Delta E)^2)]/9$. From $Q(|\Psi\rangle)=8/9$, it turns out that $|\Delta E| \approx |f'_2/3-1/2|$ should be much larger than $f'_2$. Therefore, a sufficient $\Delta f_j$ is needed to generate a $W$ state. Actually, we found that $|\Delta f_j| \approx 0.01$ is sufficient to show the generation of a $W$ state. [Figs. 3(a) and 3(b)]. In Fig. 3(b), the $W$ state is formed slightly away from the point, $f_0=f_B=0$. For a relatively weak coupling, the single-qubit tunneling $f'_2$ as well as $f'_2$ becomes larger. Thus, an additional small flux $f_2=0.0004$ will break the symmetry so that the state $|\Psi\rangle$ closer to the $W$ state would be formed. In Fig. 3(c), we can see the $W$ state around $f_0=0$. On the contrary to the $GHZ$ state where the range of $f_3$ is critical for experimental observation, for $W$ state the range in $(f_0,f_B)$ plane is important, as shown in Figs. 3(a) and 3(b). Figure 3(d) shows the $Q$ factor for $W$ states whose peak widths depend on the value of the two-qubit tunneling amplitude $f'_2$ (Table I). As $E'_j$ decreases, $f'_2$ becomes larger. However, if the coupling strength becomes too weak, the two classes with $S_z=\pm 1/2$ will become overlapped with each other through the single qubit tunneling $f'_1$ so that the $W$ state may readily be broken. Hence, as a consequence of compromise, the $W$ state emerges for an intermediate coupling.
dotted line indicates Cut view of has been done, where they simultaneously measure the state tomography measurement. Recently, for capacitively coupled phase qubits, the tomography measurement factors along dotted lines in with rather broader peak width, as shown in Table I. The quantification of entanglement can be done by using the state tomography measurement.\(^{9,25}\) Recently, for capacitively coupled phase qubits, the tomography measurement has been done,\(^{9}\) where they simultaneously measure the state of coupled qubits. For present coupling, we expect that the similar tomography measurement can also be performed.

The tripartite entanglement with superconducting qubits has not yet been achieved so far. For the bipartite entanglement, capacitively coupled phase qubits showed high fidelity in a recent experiment,\(^{9}\) while for charge qubits, only partial entanglement was observed. The interactions between phase qubits are XY-type interactions, which describe simultaneous two-qubit flipping processes. The two- or multiple-qubit tunneling processes are essential for entanglement of qubits.\(^{12}\) However, for charge qubits, the interaction is mainly Ising type. We believe that this is the reason for weak entanglement in experiments with charge qubits. For three coupled phase qubits, the lowest energy state is |000⟩ state, while the highest is |111⟩ state. Hence, the superposition between these two states will be negligibly weak, thus the GHZ state cannot be formed. However, since the other states, for example, |100⟩, |010⟩, |001⟩ states, are energetically degenerated, the W state could be obtained.

In summary, we investigate a three superconducting flux qubit system. The GHZ and W states can be realizable in the eigenstates of the macroscopic quantum system. We show that while the GHZ state needs strong coupling strength, the W state can be formed at an optimized coupling strength. Moreover, to keep the tripartite entangled states robust against external flux fluctuations for feasible experimental realizations, the three coupled qubit system can provide relatively large three-qubit and two-qubit tunneling amplitudes for GHZ and W states, respectively.

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