Development of the Bélanger Equation and Backwater Equation by Jean-Baptiste Bélanger (1828)

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A hydraulic jump is the sudden and rapid transition from a sub-critical to a subcritical flow motion, and may be considered as a flow singularity. For a horizontal rectangular channel and neglecting boundary friction, the continuity and momentum principles yield a relationship between the upstream and downstream flow depths

$$\frac{d_2}{d_1} = \frac{1}{2} \times (\sqrt{1 + 8\mathsf{F}_1^2} - 1) \tag{1}$$

where subscripts 1 and 2 refer to the upstream and downstream flow conditions, respectively; F=Froude number; $\mathsf{F}=V/\sqrt{gd}$; d and V=flow depth and velocity, respectively; and g=gravity acceleration. The hydraulic jump is typically classified in terms of its inflow Froude number $\mathsf{F}_1=V_1/\sqrt{gd_1}$, which must be greater than unity (Bélanger 1828; Henderson 1966). For F_1 slightly above unity, the hydraulic jump is characterized by a train of stationary free-surface undulations. For larger Froude numbers, the jump has a marked roller with large-scale vortices, and the flow is characterized by significant kinetic energy dissipation and air bubble entrainment. Historical contributions on the hydraulic jumps included the experiments of Bidone (1819), the theoretical analysis of Bélanger (1828, 1841), the experiments of Darcy and Bazin (1865), the solutions of Boussinesq (1877), and the work of Bakhmeteff (1932). Recent technical reviews encompass Hager (1992) and Chanson (2007, 2009).

Bélanger is commonly linked to the application of the momentum principle to the hydraulic jump (i.e., the Bélanger equation), but few people realize that his 1828 treaty was focused on the study of gradually varied open channel flows. The original work of Bélanger (1828) is reconsidered herein and it is highlighted that his development of the backwater equation was remarkable for a period when numerical integration calculations were performed by hand. Bélanger introduced the notion of critical flow conditions as a singularity of the backwater equation, and showed that the backwater equation cannot be solved across a hydraulic jump. Instead, he understood the rapidly varied nature of the jump flow, though his 1828 theoretical treatment was incorrect.

Life of Jean-Baptiste Bélanger (1790-1874)

Born in Valenciennes, in northern France, on April 4, 1790 (Chanson 2008), Jean-Baptiste Charles Joseph Bélanger studied at the Ecole Polytechnique and later at the Ecole des Ponts et Chaussées in Paris. Ingénieur du Corps des Ponts et Chaussées (Bridges and Roads Corps of Engineers), he started his engineering career in 1816. From 1821, he worked on the Somme navigation canal and from 1826 on the Ardennes navigation canal (La Houille Blanche 1960). It was during these two projects that he studied specifically the hydraulics of gradually varied open channel flows. He later became a lecturer at the Ecole Centrale des Arts et Manufactures between 1838 and 1864, at the Ecole des Ponts et Chaussées from 1841 to 1855, and at the Ecole Polytechnique from 1851 to 1860 (Chatzis 1995). At the Ecole Centrale, one of his students was Gustave Eiffel (1832–1923) who built the Eiffel tower. Jean-Baptiste Bélanger retired in 1864 and died on May 8, 1874 at Neuilly-sur-Seine.

Development of the Bélanger Equation

From 1821, Jean-Baptiste Bélanger worked as a practicing engineer on a solution of gradually varied open channel flows. He published a preliminary report in 1823 but he felt that the work lacked theoretical foundations: "il a senti de lui-même le désir de l'améliorer" ("he felt himself the need to improve it") (Bélanger 1828). His revised document was successfully examined by the Commission des Ponts et Chaussées et des Mines on 21 July 1827 and published in 1828 (Bélanger 1828).

Bélanger (1828, p. 31–36) considered the hydraulic jump as a rapidly varied flow, across which the gradually varied flow equation could not be applied (Fig. 1). Based upon the experimental observations of Bidone (1819), he treated the flow singularity by applying the energy principle, using a formulation derived from a "Traité Spécial" published in 1819 by Gustave Gaspard Coriolis (1792–1843): "je me sers du théorème de Mécanique connu sous le nom d'équation des forces vives" ("I use the mechanics theorem known as the equation of conservation of energy"). Gustave Gaspard Coriolis, another Ingénieur du Corps des Ponts et Chaussées, introduced the kinetic energy correction coefficient (Coriolis 1836) and is known for his works on rotating bodies.

Jean-Baptiste Bélanger considered the general case of a hydraulic jump in a sloping channel of irregular section. For the special case of a horizontal, rectangular, prismatic channel (Fig. 1), he derived the result

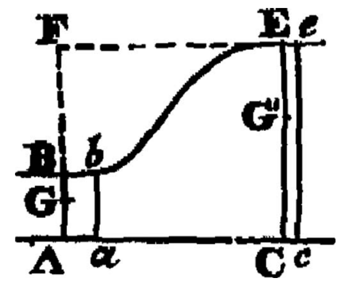

Fig. 1. Bélanger's original sketch of a hydraulic jump

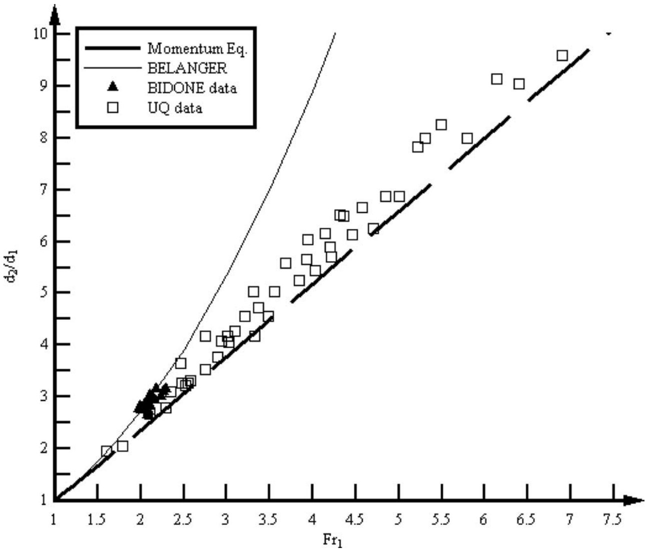

Fig. 2. Ratio of conjugate depths for a hydraulic jump in a rectangular, horizontal, and prismatic channel: comparison of Eq. (1), Eq. (3) by Bélanger (1828), experimental data by Bidone (1819), and experiments in a 0.5-*m*-wide channel at the University of Queensland

$$d_2 - d_1 = \frac{V_1^2}{2g}\left(1 - \frac{d_1^2}{d_2^2}\right) \tag{2}$$

Eq. (2) corresponds to Bélanger's Eq. (59) (Bélanger 1828, p. 35) and is nothing more than the solution of the energy equation in terms of the specific energy. It would give a reasonable approximation to the hydraulic jump solution for undular and weak jumps since there is very little energy loss in the jump for Froude numbers not much greater than unity (Montes 1986), but the development is incorrect.

Eq. (2) may be rewritten in a dimensionless form as

$$\frac{d_2}{d_1} = 1 + \frac{1}{2}\mathsf{F}_1^2\left(1 - \left(\frac{d_2}{d_1}\right)^{-2}\right) \tag{3}$$

This result, compared to Eq. (1), is obviously wrong, as illustrated in Fig. 2, because it neglects energy dissipation. While Bélanger's results matched the experimental observations for Bidone (1819) for low F_1 quite well, it diverges from the theoretical solution [Eq. (1)] at larger F_1 because energy dissipation is ignored. Fig. 2 compares Eqs. (1) and (3), and experimental observations, including data of Bidone (1819) used by Bélanger to check his results, as well as new experimental observations in the 0.5-*m*-wide rectangular channel at the University of Queensland. Simply Bélanger (1828) applied incorrectly the Bernoulli principle to the hydraulic jump.

Bélanger found his error in 1838: "de nouvelles réflexions m'ont conduit en 1838 à reconnaitre que cette hypothèse n'était pas admissible" ("new thoughts led me in 1838 to acknowledge that the assumption was incorrect") (Bélanger 1849, p. 91). Bélanger (1841) correctly solved correctly the momentum equation for a hydraulic jump in a flat channel, and his reasoning became commonly accepted thereafter (Bélanger 1849; Bresse 1860). For example, Bresse (1860, p. 251) presented the correct expression in the form

$$\frac{d_2}{d_1} = -\frac{1}{2} + \sqrt{\frac{1}{4} + 2\mathsf{F}_1^2} \tag{4}$$

that is a mere rewriting of Eq. (1).

Bélanger (1828) highlighted the significance of the inflow Froude number $\mathsf{F}_1 = V_1/\sqrt{gd_1}$, showing that a hydraulic jump occurs only for $\mathsf{F}_1 > 1$. He also showed the existence of critical flow conditions in a rectangular horizontal channel for $V^2 = gd$. This was 24 and 44 years, respectively, before the publications of Ferdinand Reech (1852) and William Froude (1872), who were both credited with the introduction of the Reech-Froude number $V/\sqrt{gd}$.

Further, Bélanger (1828) applied successfully the backwater equation upstream and downstream of the hydraulic jump, and pointed out that it cannot be applied across the jump itself. He showed also how to estimate the jump location by combining the backwater calculations, upstream and downstream of the jump, with the hydraulic jump equation.

Backwater Equation

To calculate the free-surface profiles of gradually varied open channel flows, Bélanger (1828) developed the backwater equation with the following basic assumptions: (1) a steady flow; (2) a one-dimensional flow motion; (3) a gradual variation of the wetted surface with distance x along the channel; (4) friction losses that are the same as for an uniform equilibrium flow for the same depth and discharge; and (5) a hydrostatic pressure distribution. Bélanger (1828, p. 1–11) derived the backwater equation from momentum considerations in a manner somehow similar to the modern development of normal flow conditions (Henderson 1966; Chanson 1999, 2004), obtaining

$$\sin\theta\,\partial x - \cos\theta\,\partial d - \frac{P_w}{A}(aV + bV^2) + \frac{Q^2}{gA^3}\partial A = 0 \tag{5}$$

where θ=angle between the bed and the horizontal; x=longitudinal distance positive downstream; d=flow depth measured normal to the invert; A=cross-sectional area; P_w=wetted perimeter; and Q=discharge. Eq. (5) corresponds to Eq. (16) in Bélanger (1828, p. 9). It may be rewritten in a more conventional form as a differential equation

$$\sin\theta - \cos\theta\frac{\partial d}{\partial x} - \frac{P_w}{A}(aV + bV^2) + \frac{Q^2}{gA^3}\frac{\partial A}{\partial x} = 0 \tag{6}$$

In Eqs. (5) and (6), Bélanger (1828) estimated the friction losses using the Prony formula

$$-\frac{\partial H}{\partial x} = \frac{4}{D_H}(aV + bV^2) = \frac{f}{D_H}\frac{V^2}{2g} \tag{7}$$

where H=total head; D_H=hydraulic diameter; $D_H = 4A/P_w$; and a and b are constant (a=4.44499 10^{-5} and b=3.093140 10^{-4}) (in SI units). [The values of a and b are directly reported with the same accuracy as Bélanger (1828).] In Eq. (7), the right terms correspond to the traditional expression of the head losses in terms of the Darcy-Weisbach friction factor f. Denoting S_f as the friction slope ($S_f = -\partial H / \partial x$), and S_o as the bed slope ($S_o = \sin\theta$), Bélanger's backwater Eq. (5) may be combined with the continuity equation to yield

$$\frac{\partial}{\partial x}\left(d\cos\theta + \frac{V^2}{2g}\right) = S_o - S_f \tag{8}$$

Eq. (8) is essentially identical to modern expressions of the backwater equation (Henderson 1966; Montes 1998; Chanson 2004). In its general form, Chanson (1999) expressed the backwater equation as

$$\cos\theta\frac{\partial d}{\partial x} - d\sin\theta\frac{\partial\theta}{\partial x} - \alpha\frac{Q^2}{g \times A^3}\frac{\partial A}{\partial x} = S_o - S_f \tag{9}$$

where α=kinetic energy correction coefficient, or Coriolis coefficient. The main differences between Bélanger's Eq. (8) and Eq. (9) are the Coriolis coefficient and the nonconstant bed slope term. But Bélanger made no further assumption and his development (Bélanger 1828, p. 9) is basically identical to the modern forms of the backwater equation used by today's hydraulic engineers.

Eq. (6) was tested for a nonprismatic smooth drop inlet. Fig. 3(a) shows the experimental facility and Fig. 3(b) compares the experimental observations with (a) Eq. (6) in which the flow resistance was calculated using the Prony formula (Eq. (7)), with (b) Eq. (8) in which the friction slope was calculated in terms of the Darcy friction factor, and with (c) Eq. (9). All the calculations were performed using the step-method distance calculated from depth (see following paragraphs). The experimental data (symbols [*]) are plotted together with the bed elevation z_o and sidewall profiles, and agree well with computations [Fig. 3(b)]. The results show basically very little differences between data and calculations, despite the challenging geometry and the crude nature of the Prony formula. Bélanger's calculations give identical results to modern estimates. But Bélanger had neither computer nor calculator, nor even a slide rule, to integrate the backwater equation. All the calculations were performed manually, and this explains the usage of Prony's simplified formula (Brown 2002).

Discussion

Bélanger integrated the backwater equation by selecting known water depths and calculating manually the distance in between: "il s'agit d'intégrer entre deux limites h" ("the integration takes place between two [water depth] limits h") (Bélanger 1828, p. 11–13). Today this technique is called the step-method distance calculated from depth (Henderson 1966; Chanson 1999) or the direct step method.

Further he investigated the two singularities of the backwater equation. One corresponded to the uniform equilibrium flow conditions $S_o = S_f$, for which the flow depth equals the normal depth. Bélanger (1828, p. 10) obtained the normal depth expression of Prony (1804)

$$\frac{(aV + bV^2)}{\frac{D_H}{4}} = \sin\theta \tag{10}$$

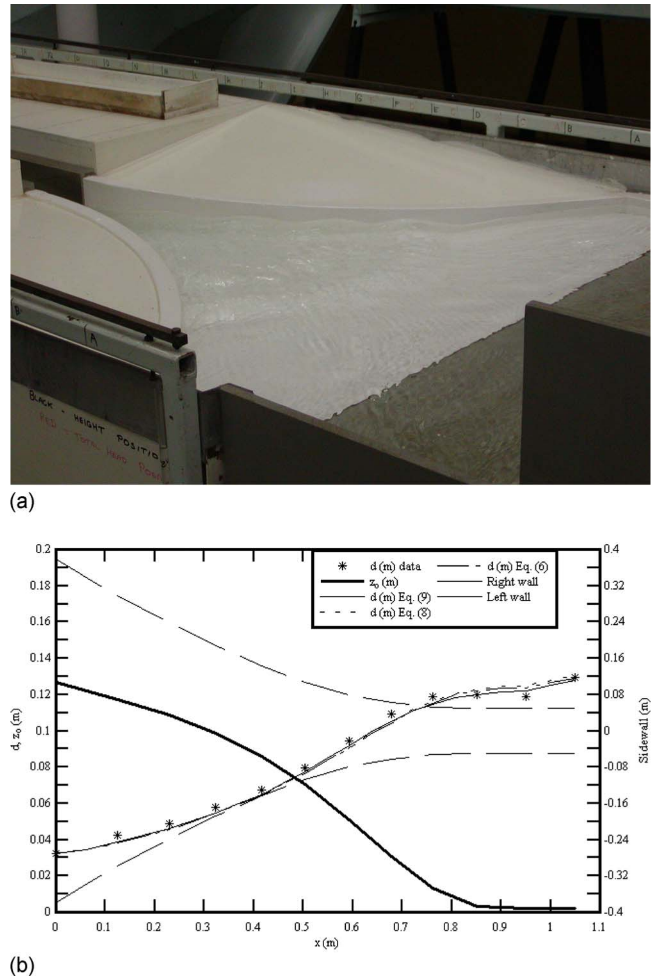

Fig. 3. Free-surface profile in a smooth drop inlet for $Q = 0.010\ m^3/s$: (a) photograph of the smooth drop inlet experiment, flow from bottom right to top left; (b) Comparison between experimental data and backwater calculations

The second singularity of the backwater equation corresponded to $\partial x / \partial d = 0$ and it yielded the condition

$$\frac{Q^2}{g\cos\theta A^3}\frac{\partial A}{\partial d} = 1 \tag{11}$$

where $\partial A / \partial d$=free-surface width. Eq. (11) expresses the critical flow conditions for a channel of irregular cross section. For a wide rectangular open channel with hydrostatic pressure distribution, it yields: $V^2 = gd\cos\theta$ (Liggett 1993; Chanson 2006). Bélanger (1828, p. 29) did not use the term "critical flow" but he highlighted explicitly the flow singularity: "un cas peu ordinaire" ("a special case"). He stressed further the physical impossibility to observe $\partial d / \partial x = +\infty$ for this special case.

Conclusion

In the 1820s, Jean-Baptiste Bélanger (1790–1874) worked on a method to calculate gradually varied open channel flow properties for steady flow conditions. Although he succeeded, his treatise (Bélanger 1828) is better known for his treatment of the stationary hydraulic jump, today called the Bélanger equation. It is shown herein that although he correctly considered a hydraulic jump as a rapidly varied flow, he applied the wrong basic principle in 1828. Bélanger applied the energy principle neglecting the rate of energy dissipation. He corrected his development 10 years later (Bélanger 1841).

The true originality of Bélanger's (1828) work lay in the successful development of the backwater equation for steady one-dimensional gradually varied flows in an open channel. His work outlined the fundamental assumptions and he derived from momentum considerations an equation that is still in use today (but for the flow resistance model). In the same study, Bélanger introduced two further modern concepts: the step-method distance calculated from depth and the critical flow conditions. He associated the notion of critical flow with one of the two singularities of the backwater equation. His technique of numerical integration was ahead of his time, particularly when there was no computer nor electronic calculator.

Considering Bélanger's contribution, the backwater equation should be called the "Bélanger equation," while the application of the momentum principle to the hydraulic jump could be referred to as the "Bélanger method."

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Notation

The following symbols are used in this paper:

A = flow cross section area (m^2);
a = coefficient of the Prony resistance formula;
B = channel width;
b = coefficient of the Prony resistance formula;
D_H = hydraulic diameter (m) defined as: $D_H=4\,A/P_w$;
d = water depth (m);
F = Froude number;
f = Darcy-Weisbach friction factor;
g = gravity acceleration (m/s^2);
H = total head (m);
P_w = wetted perimeter (m);
Q = discharge (m^3/s);
R = Reynolds number ($\mathsf{R}=\rho Vd/\mu$);
S_f = friction slope;
S_o = bed slope ($S_o=\sin\theta$);
V = flow velocity (m/s);
x = longitudinal flow direction (m);
z_o = bed elevation (m);
θ = angle between the channel bed and the horizontal;
μ = dynamic viscosity of the fluid (Pa/s); and
ρ = fluid density (m^3/s);

Subscripts:

1 = inflow conditions; and
2 = downstream conjugate flow conditions.

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