Sensitivity to measurement errors in the quantum kicked top

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We address the problem of chaos in quantum systems in terms of the sensitivity of the evolution of individual quantum states to errors in measurements. Specifically, we generate quantum trajectories of the quantum kicked top by considering discrete measurements on individual states as they evolve. If an error occurs when recording the results of the measurements we can calculate the difference between the true quantum state and the state inferred from the measurement results. Not surprisingly, the results depend on the strength of the measurement back action and also on whether the initial state is centered in a regular or chaotic region of the corresponding classical phase space. For measurements with a strong back action we find that the initial chaotic state shows sensitivity to measurement errors, but an initial regular state does not.

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I. INTRODUCTION

The question of whether or not chaos exists in quantum systems has stimulated much research in recent years, and while many avenues have been explored, each one gives a different perspective on the problem rather than definitive results. Most approaches have concentrated on finding signatures of chaos in the quantized counterpart of systems that exhibit classical chaotic dynamics [1,2]. A common approach for periodically driven systems is to study the eigenvalue spectrum for the unitary evolution operator that evolves the state over one period [3,4]. Another approach is to look at the collapse and revival sequences for operator averages [4–6].

A common feature of these methods is that they consider isolated quantum systems. Because of this, the hallmarks of classical chaos (exponential sensitivity to initial conditions as manifest in the divergence of phase-space trajectories) are not evident in the quantum systems since the unitary nature of Schrödinger evolution means that quantum states that are initially close together remain close together as they evolve (and in fact there is no divergence at all). This observation naturally leads one to consider the evolution of open quantum systems. This is important since in reality quantum systems are never isolated due to coupling with an environment or measuring apparatus, which can alter the dynamics of the system.

Early work along these lines by Sarkar and Satchell [7] and also Dittrich and Graham [8] concentrated on the quantum kicked rotor, which is chaotic in the classical limit. The quantum kicked rotor exhibits dynamic localization due to quantum interference as opposed to diffusive behavior seen in the classical system. Sarkar and Satchell [7] unexpectedly found that for a certain class of measurements there was no disruption to localization. In contrast to this, Dittrich and Graham [8], in considering the effect of continuous measurements of the action variable for the rotor, found that localization gave way to diffusion. Similarly Schlautmann and Graham [9] demonstrated the restoration of the diffusive behavior of a periodically driven pendulum in momentum space when continuous momentum measurements were performed on the system.

A general approach to this problem was taken by Zurek and Paz [10], who show how the effect of decoherence due to interaction with an environment leads to the emergence of behavior in quantum systems analogous to that in classical systems. Specifically, they find that the entropy produced in open quantum systems with chaotic classical analogs increases linearly at a rate determined by the Lyapunov exponents, whereas a regular quantum system evolves with little entropy production. As another example, Peres [11] studied the quantum evolution of a classically chaotic system, the kicked top, when the Hamiltonian is perturbed. Starting with the same initial state vector, he found that the final vector evolved under the perturbed Hamiltonian was far from the final vector evolved under the nonperturbed Hamiltonian if the initial state corresponds to a classically chaotic region of phase space, but that the two vectors remained close together if the initial state corresponds to a classically regular region of phase space. Schack et al. [12] have extended the work of Peres by using information theory to characterize the distribution of Hilbert space vectors arising from evolution under a stochastic Hamiltonian.

In this paper we also consider the evolution of an open quantum system. Specifically, we want to be able to characterize the sensitivity of the evolution of a quantum system to errors in measurements. It should be made clear here that the errors we consider in this context are not due to the intrinsic quantum fluctuations in the system or apparatus states but arise at a purely classical level due to coarse graining, or a corruption, of the final classical results. Following Schack et al. [12], we study the distribution of Hilbert space vectors, where each vector corresponds to a particular sequence of measurements made on a given initial state. We can describe the evolution of the quantum system state vector under the influence of measurements using quantum trajectories. We consider a simple measurement model in which the result of each measurement has only two possible outcomes, which we label as 0 and 1 for convenience (although “+” and “−” or “yes” and “no” can equally well be used). If a single error is made in the sequence of measurement results
on a given system, how far will an experimenter’s estimate of the quantum state be (based on the incorrect sequence of measurement results) from the true state of the system? We show that the answer to this question depends on both the type of measurements that are made and on whether or not the initial system state corresponds to a classically regular or chaotic region of phase space. By modifying the parameters of the measuring apparatus (that is, by changing the way we extract information from the environment) we can change the conditional states of the system (conditioned on the measurement results) without changing the unconditional evolution of the system. This allows us to highlight the inseparability of the definition of quantum chaos and the way the system is coupled to the environment (i.e., the type of measurements performed).

In the next section we introduce the model to be used, the quantum kicked top, and describe the measurement interactions to be considered. In Sec. III we give the results of our simulations that demonstrate the difference between regular and chaotic dynamics in terms of the distribution of Hilbert space vectors. Section IV concludes with a discussion of the relevance of the results.

II. QUANTUM KICKED TOP

A. Classical and quantum dynamics

The quantum kicked top is a nonlinear spin system that undergoes periodic kicking. The dynamics of this system has been studied extensively [4,6,11] and the corresponding classical model exhibits regular and chaotic dynamics. The Hamiltonian for the classical kicked top is given by

$$H = \frac{\kappa}{2\tau} J_z^2 + p J_y + \sum_{n=1}^{\infty} \delta(t-n\tau),$$

(1)

where \(\tau\) is the duration between kicks, \(J=(J_x, J_y, J_z)\) is the angular momentum vector, and \(j=(J, J)\) is a constant of the motion. The classical dynamics can be reduced to a two-dimensional map of points on a sphere of radius \(j\) [6] and the angular momentum vector can be parametrized in polar coordinates as

$$J = j(\sin \Theta \cos \Phi, \sin \Theta \sin \Phi, \cos \Theta).$$

(2)

The first term in the Hamiltonian (1) describes a nonlinear precession of the top about the \(z\) axis and the second term describes periodic kicks around the \(y\) axis. For the special case of \(p=\pi/2\) the classical dynamics can be described by the recursive map

$$X' = \begin{bmatrix} Z \cos \kappa X + Y \sin \kappa X \\ -Z \sin \kappa X + Y \cos \kappa X \\ -X \end{bmatrix},$$

(3)

where we have defined the normalized angular momentum variable \(X = J/j\). In terms of \(X\), the dynamics can be mapped onto a unit sphere. In Fig. 1 we show the classical map for \(p=\pi/2\) and \(\kappa=3\). For these parameters it can be seen that the classical map has well defined regular and chaotic regions.

![Phase-space portrait for the classical map of the quantum kicked top](image)

FIG. 1. Phase-space portrait for the classical map of the quantum kicked top, with \(p=\pi/2\) and \(k=3\). The lower circle is the projection onto the X-Z plane for the northern hemisphere \((Y>0)\), while the upper circle is for the southern hemisphere. A righthand orientation of the X, Y, and Z axes is maintained. The points corresponding to the centers of the initial spin coherent states are denoted by a filled diamond (|\(\psi_R\rangle\)), a filled square (|\(\psi_C\rangle\)), and a filled circle (|\(\psi_C^2\rangle\)).

We specify the quantum map corresponding to Eq. (3) by the unitary evolution operator

$$U = \exp \left(-i \frac{3}{2j} J_z^2 \right) \exp \left(-i \frac{\pi}{2} J_y \right),$$

(4)

which takes a state from just before one kick to just before the next, i.e., \(|\psi\rangle \rightarrow U |\psi\rangle\). where \(J_z\) and \(J_y\) are the usual angular momentum operators and \(j\) is the angular momentum quantum number. The first exponential \(U_p = \exp (-i(3/2j)J_z^2)\) describes the precession about the \(z\) axis, and the second \(U_K = \exp (-i(\pi/2)J_y)\) describes the kick.

For a given value of \(j\), the Hilbert space for the kicked top has dimension \(2j+1\). Given the form of the evolution operator, it is most convenient to work in the basis of orthonormal \(J_z\) eigenstates \(|j,m\rangle; -j \leq m \leq j\), which satisfy \(J_z |j,m\rangle = m |j,m\rangle\) and \(J_y |j,m\rangle = j(j+1) |j,m\rangle\). For initial states we use spin coherent states, which can be written in the form [1]...
\[ \langle jm | \Theta | \Phi \rangle = (1 + \gamma \gamma^*)^{-i} \gamma^{j-m} \sqrt{\frac{2j}{j-m}}, \]  

(5)

where \( \gamma = e^{i\Phi \tan(\Theta/2)} \). For our simulations we choose initial states centered in both the regular and chaotic regions of the classical phase space. Following Sanders and Milburn [6], for a state in the regular region, denoted \( | \psi_R \rangle \), we use \( \Theta = 2.25 \) and \( \Phi = 0.63 \). For the chaotic region we pick two initial states. The first one is the same as in Ref. [6], which we shall denote \( | \psi_{C1} \rangle \) and for which \( \Theta = 1.64 \) and \( \Phi = 1.50 \). For the second state in a chaotic region \( | \psi_{C2} \rangle \), we use \( \Theta = 0.89 \) and \( \Phi = 0.63 \). The state \( | \psi_{C2} \rangle \) is chosen to have the same \( J_z \) distribution as \( | \psi_R \rangle \). In Fig. 1 we plot the points corresponding to the centers of these initial states.

**B. Measurement model**

To describe measurements on the kicked top we follow the general formalism of quantum measurements based on operations and effects [13–16]. Consider a composite system comprised of a system under study and a meter. We assume at time \( t \) we can write the joint state as a product \( \rho(t) = \rho_S(t) \otimes \rho_M(t) \), where \( \rho_S(t) \) and \( \rho_M(t) \) are the states of the system and meter, respectively. In order to perform a measurement we allow the system and meter to interact for a time \( \tau \), which we assume is small compared to the typical times associated with the free dynamics of the system and meter. After the measurement interaction the composite system is in the state

\[ \rho(t + \tau) = U_I(\tau) \rho_S(t) \otimes \rho_M(t) U_I^\dagger(\tau), \]  

(6)

where \( U_I(\tau) \) is the evolution operator describing the interaction. The coupling between the system and meter produces an entangled state of the two systems. After the measurement interaction is complete we consider a projective measurement on the meter, which disentangles the system and meter. The state of the composite system conditioned on the measurement result \( \alpha \) is then

\[ \tilde{\rho}^\alpha(t + \tau) = P_\alpha U_I(\tau) \rho_S(t) \otimes \rho_M(t) U_I^\dagger(\tau) P_\alpha, \]  

(7)

where \( P_\alpha = |\alpha\rangle \langle \alpha | \) is the projector onto the meter state \( |\alpha\rangle \) and the tilde indicates the density matrix has a nonunit trace (i.e., the density operator is left unnormalized). By tracing over the meter states we obtain the conditional system state

\[ \tilde{\rho}_S^\alpha(t + \tau) = \langle \alpha | U_I(\tau) \rho_S(t) \otimes \rho_M(t) U_I^\dagger(\tau) | \alpha \rangle. \]  

(8)

If the initial meter state is given by \( \rho_M(t) = \sum_j p_j | \phi_j \rangle_M \langle \phi_j | \), we find

\[ \tilde{\rho}_S^\alpha(t + \tau) = \sum_j \Omega_{\alpha,j}(\tau) \rho_S(t) \Omega_{\alpha,j}^\dagger(\tau), \]  

(9)

where the measurement operators \( \Omega_{\alpha,j}(\tau) \) are given by

\[ \Omega_{\alpha,j}(\tau) = \sqrt{p_j} \langle \alpha | U_I(\tau) | \phi_j \rangle_M. \]  

(10)

The probability that the system will be in the state \( \tilde{\rho}_S^\alpha(t + \tau) \) is

\[ p_\alpha = \text{tr}_S[\tilde{\rho}_S^\alpha(t + \tau)] = \text{tr}_S \left[ \sum_j \Omega_{\alpha,j}(\tau) \rho_S(t) \Omega_{\alpha,j}^\dagger(\tau) \right], \]  

(11)

where the trace is over the system Hilbert space.

The conditional density operator given above gives the state of the system given complete knowledge about the measurement outcomes. It may turn out, however, that these results are not known or indeed are unknowable for the reason that the meter states that record them are such complicated environmental states that to collect complete information is impossible. If the results are indeed unknown, we can then describe the system by an unconditional density operator by averaging over the possible results of the measurement. This unconditional density operator is then given by

\[ \rho_S(t + \tau) = \sum_\alpha \sum_j \Omega_{\alpha,j}(\tau) \rho_S(t) \Omega_{\alpha,j}^\dagger(\tau). \]  

(12)

It should be noted, however, that there may be many ways of constructing conditional density operators corresponding to the same unconditional density operator. The importance of this statement in the context of this paper will become clearer below.

Turning now to the specific measurement model used, we take as our meter a spin-\( \frac{1}{2} \) system that is coupled to the \( y \) component of spin of the kicked top via the interaction

\[ U_I = \exp(-i\chi I_z s_z), \]  

(13)

where \( \chi \) determines the strength of the interaction (which depends on the interaction time) and \( s_z \) is the spin angular momentum operator for the meter in the \( z \) direction. This two-valued measurement model is clearly not the most general that we could choose to make on a kicked top. Indeed, using a meter with only two possible outcomes is ill matched to the Hilbert space dimensions of a top with large angular momentum. However, a meter with two distinct meter states is the minimum number of outcomes required to make any measurement at all. We choose to begin with these simple yes/no measurements for reasons of simplicity and computational convenience. In addition, more complicated measurement models can always be reduced to simple yes/no measurements by coarse graining the outcomes. We will leave consideration of more general apparatus models for further work and return now to a description of a two-dimensional measurement apparatus for the kicked top.

We define orthogonal projectors \( P_\pm \) on the meter Hilbert space as

\[ P_\pm = |n_\pm \rangle \langle n_\pm |, \]  

(14)

where

\[ |n_+ \rangle = \cos \theta \left| \frac{1}{2} \right>_z + \sin \theta e^{i\phi} \left| - \frac{1}{2} \right>_z, \]  

(15)

\[ |n_- \rangle = \cos \theta \left| - \frac{1}{2} \right>_z - \sin \theta e^{-i\phi} \left| \frac{1}{2} \right>_z, \]  

(16)
and \( |\pm \frac{1}{2}\rangle_c \) are the eigenstates of \( s_z \) that satisfy \( s_z|\pm \frac{1}{2}\rangle_c = \pm \frac{1}{2}|\pm \frac{1}{2}\rangle_c \). By choosing different values of \( \theta \) and \( \alpha \) we can project onto any combination of the spin eigenstates of the meter. Because the meter is a spin-\( \frac{1}{2} \) system there are only two possible results for each measurement, corresponding to the meter being projected onto \( |n_+\rangle \) or \( |n_-\rangle \). For ease of notation we label the + result 1 and the – result 0. We can then represent the results of a sequence of measurements, performed on the same system, by a binary string, with each digit corresponding to the result of a single measurement. The simplest error in the measurement record will then be a bit error, where one digit in the string is incorrect. We consider two distinct cases.

Case 1. We choose \( \theta = 0 \), which corresponds to a projective measurement onto \( |\pm \frac{1}{2}\rangle_c \), and we assume that before each measurement the meter is prepared in the mixed state \( \rho_M(t) = \frac{1}{2}(|\frac{1}{2}\rangle_c\langle \frac{1}{2}| + |\frac{1}{2}\rangle_c\langle -\frac{1}{2}| - |\frac{1}{2}\rangle_c\langle \frac{1}{2}| - |\frac{1}{2}\rangle_c\langle -\frac{1}{2}| \). Then, if the system is in the pure state \( \rho_S(t) = |\psi\rangle\langle \psi| \) before the measurement it is easy to show from Eq. (9) that the unnormalized state after the measurement is also pure and is given by

\[
|\tilde{\psi}^\pm\rangle = \frac{1}{\sqrt{2}} \exp\left( \mp \frac{i}{2} \frac{\chi}{j} J_y \right) |\psi\rangle,
\]

where the superscript + or – indicates the measurement result upon which the state condition is. Clearly the probability for both results is equal to 1/2, regardless of the system state. In fact, to call this case a measurement is a bit misleading as the measurement results are simply given by a coin toss and hence we gain no knowledge of the system state from the measurement results. However, the effect on the system state corresponds precisely to the random perturbations introduced by Schack et al. [12] and we include it for comparison.

Case 2. We choose \( \theta = \pi/4 \) and \( \alpha = \pi/2 \), corresponding to a projection onto \( |\pm \frac{1}{2}\rangle_c \), and we assume the meter is in the pure state \( \rho_M(t) = |\frac{1}{2}\rangle_c\langle \frac{1}{2}| + |\frac{1}{2}\rangle_c\langle -\frac{1}{2}| \). Again, if the system is in the pure state \( |\psi\rangle \) we can show the postmeasurement states are given by

\[
|\tilde{\psi}^+\rangle = \cos \left( \frac{\chi}{2} J_y \right) |\psi\rangle,
\]

\[
|\tilde{\psi}^-\rangle = \sin \left( \frac{\chi}{2} J_y \right) |\psi\rangle.
\]

The probabilities for the system to be in the states \( |\tilde{\psi}^+\rangle \) and \( |\tilde{\psi}^-\rangle \) are \( p_+ = \langle \psi| \cos^2(\chi/2 J_y) |\psi\rangle \) and \( p_- = \langle \psi| \sin^2(\chi/2 J_y) |\psi\rangle \), respectively. In contrast to case 1, information about the system state (specifically, the mean and variance of \( J_y \)) can in principle be obtained from knowledge of the measurement result probabilities [17].

The unconditional density operators describing the postmeasurement state for both measurement models are identical and given by the transformation

\[
\rho_S - \rho_S' = \frac{1}{2} e^{-i(\chi/2) J_y} \rho_S e^{i(\chi/2) J_y} + \frac{1}{2} e^{i(\chi/2) J_y} \rho_S e^{-i(\chi/2) J_y}.
\]

However, while the unconditional postmeasurement states are the same, we will see that the behavior of the conditional states differs markedly between cases 1 and 2. For small \( \chi \) the unconditional density operator may be written as

\[
\rho'_S = \rho_S - \frac{\chi^2}{8} [J_y, [J_y, \rho_S]] + \cdots.
\]

The double commutator has two important and complementary effects. First, it leads to decoherence in the basis of \( J_y \). To see this we take the matrix elements of \( \rho'_S \) in this basis

\[
\langle m| \rho'_S |n\rangle = \left( 1 - \frac{\chi^2}{8} (n-m)^2 \right) \langle m| \rho_S |n\rangle,
\]

where \( J_y |m\rangle = |m\rangle \). This indicates that superpositions of states widely separated in this basis are strongly suppressed, which is what one expects for a measurement of \( J_y \). Second, this term drives a diffusion process in the complementary variable. To see this we formally define the “phase states” for \( J_y \) by

\[
|\theta\rangle = \frac{1}{2j+1} \sum_{n=-j}^j e^{in\theta} |m\rangle.
\]

In this basis we find

\[
\langle \theta| \rho'_S |\theta\rangle = \left( 1 + \frac{\chi^2}{2} \frac{\partial^2}{\partial \theta^2} \right) \langle \theta| \rho_S |\theta\rangle.
\]

The second-order derivative defines a diffusion process for the variable \( \theta \). The geometric interpretation of this is as follows. We represent an eigenstate \( |m\rangle \) of \( J_y \) by a band on a sphere of radius approximately given by \( j \). The band is orthogonal to the \( J_z \) direction and has a projection onto the \( J_y \) axis of \( m \). In the absence of any other dynamics, a state localized on this band will undergo a slow diffusion along it due to the double commutator term. The interplay between diffusion and nonlinear dynamics in open chaotic systems has been discussed by Zurek and Paz [10]. In our model the phase diffusion prevents the nonlinear chaotic dynamics from producing states concentrated on too narrow a region in the spherical phase space.

III. NUMERICAL RESULTS

To perform the numerical simulations for this system, we choose \( j = 18 \) for the angular momentum quantum number and \( \chi = 0.006 \) for the measurement coupling constant. Starting with an initial coherent state as defined in Eq. (5), each trajectory was generated over a set number of intervals, each interval consisting of the following steps.

Step 1. A kick was applied to the state via the operator \( U_K = \exp[-i(\pi/2) J_y] \).
Step 2. A measurement was performed on the state. This involved transforming the state according to Eq. (17) for case 1, or Eq. (18) or (19) for case 2, depending on the measurement result, followed by normalization.

Step 3. The state was evolved over the period between kicks using the operator $U_P = \exp[-i(3/2j)J_y^2]$.

To determine which measurement operator is to be applied at step 2 in each interval it is easiest to first fix the number of intervals $N$ and then generate a set of binary strings of length $N$, each representing a record of measurement results. For each trajectory a particular measurement record is sequentially read. If a ‘‘1’’ is read the state undergoes the transformation $|\psi\rangle \rightarrow |\tilde{\psi}^+\rangle$, whereas if a ‘‘0’’ is read the state is transformed to $|\tilde{\psi}^-\rangle$.

For case 1 each trajectory evolves over 12 steps. We can easily generate all $2^{12}$ binary strings representing all possible measurement records, and using these we can calculate all $2^{12}$ trajectories. As all states are equally likely (since the probability for both the + and − measurement results are equal to $1/2$), the set of measurement records generated this way is a fair representation of the results that would be obtained in an experiment. As a measure of the distribution of the state vectors in Hilbert space we can calculate the average angle between all pairs of vectors. That is, we calculate the quantity

$$\bar{\theta} = N_1 \sum_{i=1}^{2^{N-1}} \sum_{j=i+1}^{2^{N}} \theta_{i,j},$$

where $\bar{\theta}$ is the average angle, $2^N$ is the number of trajectories/state vectors, $\theta_{i,j} = \cos^{-1}[\langle \psi_i | \psi_j \rangle]$ is the angle between the two state vectors $|\psi_i\rangle$ and $|\psi_j\rangle$, and $N_1 = 1/(2^{2N-1} - 2^{N-1})$ is a normalization constant. In Fig. 2(a) we plot $\bar{\theta}$ for the three initial states evolved up to 12 steps. As can be seen, the average angle between vectors starting from the initial regular state is less than for the two chaotic states, as expected. This tells us that the conditional states evolved from the initial regular state explore less of the available Hilbert space than the states evolved from the initial chaotic states, which is consistent with the results of Schack et al. [12]. Due to restrictions on computational resources, it is not possible to store and manipulate more than about $2^{12}$ vectors. However, we can get an approximate picture of how the vectors are distributed after a large number of steps if we randomly sample $2^{12}$ binary strings from the possible set of strings and calculate the corresponding states.

In Fig. 2(b) we plot $\bar{\theta}$ for states evolved up to 450 steps by randomly sampling $2^{12}$ of the possible $2^{450}$ measurement records. For all three initial states the vectors still appear to be diverging, although the rate of divergence is decreasing. As noted in the Introduction, we want to be able to characterize the quantum dynamics in a way that is meaningful to an experimenter. Clearly the distribution of vectors in Hilbert space tells us something about the dynamics, but the angle between Hilbert space vectors is not readily measurable. Consider an experiment that involves starting with a particular quantum state that is allowed to evolve according to some Hamiltonian and upon which measurements with binary results are periodically made. Suppose that the measurements are not perfect and that the probability for an error means that in a typical run of the experiment with $N$ measurement results one of the $N$ results will be incorrect. That is, the experimenter records the result 1, say, when the true result is 0. The experimenter believes the state is transformed according to $|\psi\rangle \rightarrow |\tilde{\psi}^+\rangle$ as a result of the measurement, when $|\psi\rangle \rightarrow |\tilde{\psi}^-\rangle$ is the actual transformation. Now consider the question, how far is the experimenter’s estimate of the quantum state from the true quantum state? To answer this question we generate pairs of measurement records in which the members of each pair differ in only one place. For each pair we can calculate the corresponding conditional state vector and hence the angle between the vectors in each pair. Figure 3 is a plot of the average of the angles between $10^5$ pairs of such vectors. This graph shows us directly how far off the experimenter’s estimate will be in terms of the angle between the true and inferred states. As can be seen, the estimate for the initial regular case is much better than for the chaotic cases, although in all cases the angles are quite small. Interestingly in all three cases the curves reach a plateau after about 30 steps, even though Fig. 2(b) shows that the average angle between all vectors is still increasing after 450 steps.

Turning now to case 2, the situation is different because the measurement records and hence the corresponding conditional state vectors are not equally probable. In fact, for small coupling constant $\chi$, the + result has a high probability $p_+ \approx 1 - (\chi^2/4)\langle J_y^2 \rangle$, while $p_- \approx (\chi^2/4)\langle J_y^2 \rangle$. We can generate all possible binary strings, but this would represent an unlikely sampling of results that would be obtained in a series of experimental runs. However, we can account for the nonuniform distribution of measurement records and corresponding conditional states by weighting with the actual
However, as we will discuss in the next section, there is a striking difference between the two cases. In Fig. 4 we see that the curve for the chaotic case tends to \( \pi/2 \), indicating that measurement errors have large effects on the predictability of the evolution of the state.

**IV. DISCUSSION**

Let us first discuss case 2. What are the practical consequences of the results given above? To answer this consider what happens if the measurement record becomes corrupted, perhaps due to classical noise in the meter. As discussed above, this could result in a bit error in the binary string encoding the measurement records. If such an error occurs, the inferred conditional state of the system will also be in error. Future measurement results may then appear to be inconsistent with the inferred conditional states and in an extreme case may be essentially random and independent of the inferred conditional state. As can be seen in Fig. 4, if the system started in the regular region, such a bit error in the record will not be a serious problem as conditional states tend to remain close together in Hilbert space, at least for the most likely measurement records. However, for the initial state \( |\psi_{C2}\rangle \) an error will lead to an inferred state that is likely to be almost orthogonal to the correct state at that time. Subsequent predictions made with this state could then be very different from the actual measured results. In this way we see that for a realistic meter it will be much more difficult to track a system state if it starts in the chaotic region than if it starts in the regular region. The best strategy under these circumstances would be simply to abandon trying to track the state and predict the measurement results and simply average over the measurement records by using the unconditional state at any time for future predictions. It is also interesting to note that for the initial state \( |\psi_{C1}\rangle \) the average angle does not approach \( \pi/2 \), but is actually close to that found for the regular initial state. The reason for this is that \( |\psi_{C1}\rangle \) has support on only a small number of Floquet states (i.e., eigenstates of the evolution operator for the quantum map), even though it is centered in a classically chaotic region. Because of this it behaves similarly to the regular state, which has support on even fewer Floquet states [6]. This is in contrast to \( |\psi_{C2}\rangle \), which has support on a large number of Floquet states.

Consider now case 1, for which Fig. 3 shows the average angle between vectors whose measurement records differ by one bit. Again the states that are initially localized in the chaotic region become separated more than for the initial regular states, but even for the chaotic case the average angle between vectors is less than 0.006 rad, which is very small compared to case 2, where the average angle for the chaotic state was close to \( \pi/2 \).

Clearly the way we make measurements on the system strongly affects our predictive power regarding the evolution of the state when our measurements are imperfect, even though the unconditional state is the same in both cases 1 and 2. For case 2 errors in the measurements lead to larger errors in our predictions than in case 1. Case 2 shows sensitivity to measurement errors when the initial state corresponds to a classical region of chaos in the phase space, which may be thought of as analogous behavior to sensitivity to initial conditions seen in classically chaotic systems. This
is in contrast to case 1, where measurement errors do not lead to large errors in the prediction of the future evolution of the state, irrespective of the initial state. These results are not surprising since one of the features that distinguishes quantum from classical systems is that in the quantum case we cannot separate the system under study from the process of measurement. When we change the way we make measurements, the composite system comprised of the system under study and the meter is changed, so we expect different results. The reason for the difference here is that in case 1 the effect of the measurement is to cause a small unitary perturbation to the state of the form \( \exp \left( \frac{i}{2} \right) \), whereas in case 2 there is a much stronger back action due to the measurement, as evidenced by the nonunitary cosine and sine form of the measurement operators.

Another interesting point is that we can choose different combinations of initial meter state and meter projection operators to give the same conditional evolution. For example, if we make the choice \( \rho_M(t) = \frac{1}{2} \left| + \frac{1}{2} \right\rangle \langle + \frac{1}{2} \right| \) for the initial meter state and project the meter onto the \( z \) direction we get the same results as for case 1 above (i.e., we find the same conditional states).

In summary, we have demonstrated that for the quantum kicked top with a given fixed unconditional evolution due to measurements, it is possible to unravel the evolution in two ways and that the behavior of the conditional states for each unraveling leads to different conclusions about the sensitivity of the system to measurement errors. To generalize this result we need to consider other systems, as well as continuous measurements. We expect in general that when the measurement has a strong (nonunitary) back action on the system it will show sensitivity to measurement errors for initial states corresponding to classically chaotic regions of phase space.