A half-car model for dynamic analysis of vehicles with random parameters

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Abstract: A half-car model is used to investigate the dynamic response of cars with uncertainty under random road input excitations in this paper. The mass of the vehicle body, mass moment of inertia of the vehicle body, masses of the front/rear wheels, damping coefficients and spring stiffness of front/rear suspensions, distances of the front/rear suspension locations to the centre of gravity of the vehicle body and the stiffness of front/rear tires are considered as random variables. The road irregularity is considered a Gaussian random process and modeled by means of a simple exponential power spectral density. The mean value and standard deviation of the vehicle’s natural frequencies are obtained by using the Monte-Carlo simulation method. The influences of the randomness of the vehicle’s parameters on the vehicle’s dynamic characteristic and response are investigated in detail using a practical example.

Keywords: stochastic half car model, vehicle, random variables, random response.

1 Introduction

The vibration of an on-road vehicle is predominantly excited by the unevenness of the road surface on which the vehicle travels. Vehicle dynamic analysis has been a hot research topic for many years due to its important role in ride comfort, vehicle safety and overall vehicle performance. Numerous papers about the theoretical and experimental investigation on the dynamic behaviour of passively and actively suspended road vehicles have been published [1-3]. The quarter-car model [4], half-car model [5] and full-vehicle model [6] have been developed with researches related to the dynamic behaviour of vehicle and its vibration control.

Although mathematical modelling tools for analysis/computation have experienced a tremendous growth, most research in vehicle dynamics was based on the assumption that all parameters of vehicle systems are deterministic. Actually, the spring stiffness and damping rate may vary with respect to the nominal value due to production tolerances and/or wear, ageing... etc. The vehicle body mass and the tyre radial stiffness can have stochastic variations due to the variety of possible vehicle loading conditions and to the uncertainty of the inflating pressure of poorly maintained tyres [7]. In cars and buses, weight and placements of passengers often exhibit significant variability. In addition, even same brand and type vehicles leaving the production line may have uncertainties in size, mass and performance and so on. Hence, the problem of vehicle vibration subject to uncertain parameters is of great significance in realistic engineering applications.

In this paper, a four-degree-of-freedom half car model is used to investigate the vibration response of cars with uncertainty under random road input excitations. The vehicle’s parameters are considered as random variables and the road unevenness is considered a Gaussian random process and modeled by means of a simple exponential power spectral density (PSD), the so-called “one slope PSD”. The first two statistical moments of the dynamic characteristic and response are obtained by using conventional Monte-Carlo simulation method. A practical example is used to investigate the influences of the uncertainty of the vehicle’s parameters on the vehicle’s dynamic behaviour.

2 Vehicle model and dynamic analysis

Consider the model of a passenger car subjected to irregular excitation from a road surface as shown in Figure 1. The equations of motion for the vehicle body and the front/rear wheels are given by

\[ m_x \ddot{x}_x + c_x (\ddot{x}_x - \ddot{x}_u) + c_{x1} (\dddot{x}_x - \dddot{x}_u) + k_x (x_x - x_u) + k_{x1} (x_x - x_u) = 0 \]  \hspace{1cm} (1)

\[ I_x \ddot{\theta}_x + I_x_c (\dot{x}_x - \dot{x}_u) + I_x_k (x_x - x_u) - I_x (\ddot{x}_x - \ddot{x}_u) = 0 \]  \hspace{1cm} (2)

\[ m_{x1} \ddot{x}_{x1} - c_{x1} (\dddot{x}_x - \dddot{x}_u) - k_{x1} (x_x - x_u) + k_{x1} (x_x - x_u) = 0 \]  \hspace{1cm} (3)
The displacement $x_1$ and $x_2$ may be represented by a random variable defined by a stationary and ergodic stochastic process with zero mean value. The power spectral density of the process may be determined on the basis of experimental measurements and in the literature there are many different formulations for it. In this paper for sake of simplicity, the following spectrum [7] is considered
\[ S_{r_i}(\omega) = S_{r_{i'}}(\omega) = \frac{(A_i v)}{\omega^2} \tag{9} \]

From equations (8) and (9), the power spectral density matrix \([S_r(\omega)]\) of \(\{P\}\) can be obtained

\[
[S_r(\omega)] = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & k_i^2 \frac{A_i v}{\omega^2} & 0 \\
0 & 0 & k_i^2 \frac{A_i v}{\omega^2} & 0
\end{bmatrix}
\tag{10}
\]

Equation (7) presents a set of coupled differential equations. If the vehicle is initially considered at rest, then its solution can be obtained in terms of the decoupling transform and Duhamel integral \([9]\)

\[
[u(t)] = \int_0^t [h(t-\tau)] [b(\tau)] [P(\tau)] d\tau
\tag{11}
\]

where \([b]\) is the normal modal matrix of the vehicle. \([h(t)]\) is the impulse response function matrix of the vehicle, and can be expressed as

\[
[h(t)] = \text{diag}[h_j(t)], \quad h_j(t) = \begin{cases} \frac{1}{\omega_j} \exp(-\zeta_j \omega_j t) \sin \omega_j t & t \geq 0 \\ 0 & t < 0 \end{cases}, \quad j = 1,2,3,4
\tag{12}
\]

where \(\omega_j\) and \(\zeta_j\) are respectively the \(j^{th}\) natural frequency and modal damping of the vehicle, and \(\omega_{j'} = \omega_j (1-\zeta_j^2)^{1/2}\).

Using Rayleigh’s quotient, the \(j^{th}\) natural frequency can be expressed as

\[
\omega_j^2 = \langle b_j | [K] | b_j \rangle / \langle b_j | [M] | b_j \rangle
\tag{13}
\]

\[
\zeta_j = \langle b_j | [C] | b_j \rangle / (2 \omega_j \langle b_j | [M] | b_j \rangle)
\tag{14}
\]

From equation (11), the correlation function matrix of the displacement response of the vehicle

\[
[R_e(\varepsilon)] = E\{u(t)u(t+\varepsilon)\} = \int_0^\infty [b] [h(t)] [b]^T [R_p(t-\tau_1+\varepsilon)] [h^T] [h(t)] [b]^T d\tau_1 d\tau
\tag{15}
\]

where \([R_p(t-\tau_1+\varepsilon)]\) is the correlation function matrix of the \(\{P(t)\}\). By performing a \([R_e(\varepsilon)]\) Fourier transformation, the power spectral density matrix of the displacement response \([S_r(\omega)]\) is

\[
[S_r(\omega)] = [b] [H(\omega)] [b]^T [S_p(\omega)] [b]^T [H^*(\omega)] [b] \tag{16}
\]

where \([H^*(\omega)]\) is the conjugate matrix of \([H(\omega)]\), \([H(\omega)]\) is the frequency response function matrix of the vehicle and can be expressed as

\[
[H(\omega)] = \text{diag}[H_j(\omega)], \quad H_j(\omega) = \frac{1}{\omega_j^2 - \omega^2 + i \cdot 2 \zeta_j \omega_j \omega}, \quad i = \sqrt{-1}, \quad j = 1,2,3,4
\tag{17}
\]

Integrating \([S_r(\omega)]\) within the frequency domain, the mean square value matrix of the vehicle’s displacement response, that is, \([y_c^2]\) can be obtained

\[
y_c^2 = \int_0^\infty [S_r(\omega)] d\omega = \int_0^\infty [b] [H(\omega)] [b]^T [S_p(\omega)] [b]^T [H^*(\omega)] [b] d\omega
\tag{18}
\]
\[ y^2 \] is a $4 \times 4$ matrix and can be expressed as
\[
\begin{bmatrix}
\psi_{a1}^2 & \psi_{a2}^2 & \psi_{a3}^2 & \psi_{a4}^2 \\
\psi_{b1}^2 & \psi_{b2}^2 & \psi_{b3}^2 & \psi_{b4}^2 \\
\psi_{c1}^2 & \psi_{c2}^2 & \psi_{c3}^2 & \psi_{c4}^2 \\
\psi_{d1}^2 & \psi_{d2}^2 & \psi_{d3}^2 & \psi_{d4}^2 \\
\end{bmatrix}
\]

(19)

From equations (5), (6) and (18), the mean square values of $m$, and $\theta$, can be respectively obtained as follows
\[
\psi_{m}^2 = (l^2 \psi_{a1}^2 + l^2 \psi_{a3}^2 + l^2 \psi_{a3}^2 + l^2 \psi_{a3}^2) / l^2
\]
(20)
\[
\psi_{\theta}^2 = (l^2 \psi_{a1}^2 - \psi_{a3}^2 - \psi_{a3}^2 + \psi_{a3}^2) / l^2
\]
(21)

3 random response analysis of vehicle with uncertain parameters

The vehicle's parameters corresponding to $m$, $l_s$, $m_o$, $c_1$, $c_2$, $k_1$, $k_2$, $l_1$, and $l_2$ are simultaneously considered as random variables. The randomness of vehicle's parameters will result in randomness of the matrices $[M]$ and $[K]$ and $[C]$, and consequently the natural frequencies $\omega_j$, modal matrix $[\phi]$, and modal damping $\zeta_j$. The random variables are each given a mean value ($\mu$) and standard deviation ($\sigma$), for example, $m_i = \mu_{mi} \pm \sigma_{mi}$. A further parameter used in this paper is the variation coefficient $\nu$, defined by the ratio of the standard deviation to the mean value, that is $\nu = \sigma / \mu$.

In the MCSM, N samples of the random variables are generated in given ranges. The implementation of the method consists in the numerical simulation of these samples associated to the random quantities of the physical problem, the procedure used for a deterministic analysis is repeated for each sample of the simulation process, obtaining then N responses that are computed to get the first two statistical moments of the response. For the four-degree-freedom system, the computational effort is acceptable for analysis of the mean value and standard deviation of vehicle's dynamic characteristics and random response.

4 Numerical examples

The mean values of vehicle's parameters for this study are given in Table 1 [11]. In the following simulations, $A_i = 1.4 \times 10^{-9}$ and $v = 50$ (m/s) are taken into consideration. In order to investigate the effect of the uncertainty of random variables $m_s$, $l_s$, $m_o$, $c_1$, $c_2$, $k_1$, $k_2$, $l_1$, and $l_2$ on the vehicle's dynamic characteristics and responses, the values of their variation coefficients $\nu_{m_i}$, $\nu_{l_i}$, $\nu_{c_i}$, $\nu_{k_i}$, $\nu_{l_i}$, and $\nu_{l_i}$ are respectively taken as different groups. The computational results of natural frequencies and mean square responses are respectively given in Tables 2 and 3, in which 10000 simulations are used. In these tables, symbol $\nu$ denotes $\nu_{m_i} = \nu_{l_i} = \nu_{c_i} = \nu_{k_i} = \nu_{l_i} = \nu_{l_i}$.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Mean values</th>
<th>Parameters</th>
<th>Mean values</th>
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</thead>
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<td>$m_s$</td>
<td>1794.4 kg</td>
<td>$k_{s1}$</td>
<td>66824.4 N/m</td>
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<tr>
<td>$l_s$</td>
<td>3443.05 kgm$^2$</td>
<td>$k_{s2}$</td>
<td>18615.0 N/m</td>
</tr>
<tr>
<td>$m_o$</td>
<td>87.15 kg</td>
<td>$c_{s1}$</td>
<td>1190 Ns/m</td>
</tr>
<tr>
<td>$c_{s2}$</td>
<td>140.4 kg</td>
<td>$c_{t1}$</td>
<td>1000 Ns/m</td>
</tr>
<tr>
<td>$l_1$</td>
<td>1.271 m</td>
<td>$k_{t1}$</td>
<td>101115.0 N/m</td>
</tr>
<tr>
<td>$l_2$</td>
<td>1.716 m</td>
<td>$k_{t2}$</td>
<td>101115.0 N/m</td>
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Table 2. The computational results of natural frequencies (unit: rad/s)

<table>
<thead>
<tr>
<th>Model</th>
<th>$\mu_{\omega_1}$</th>
<th>$\sigma_{\omega_1}$</th>
<th>$\mu_{\omega_2}$</th>
<th>$\sigma_{\omega_2}$</th>
<th>$\mu_{\omega_3}$</th>
<th>$\sigma_{\omega_3}$</th>
<th>$\mu_{\omega_4}$</th>
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<td>4.6806</td>
<td>0</td>
<td>6.3951</td>
<td>0</td>
<td>29.2741</td>
<td>0</td>
<td>44.2143</td>
<td>0</td>
</tr>
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<td>4.4124</td>
<td>1.5346</td>
<td>6.4156</td>
<td>0.1570</td>
<td>29.2744</td>
<td>0.0030</td>
<td>44.2159</td>
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<td>6.4191</td>
<td>0.1757</td>
<td>29.2746</td>
<td>0.0046</td>
<td>44.2158</td>
<td>0.0153</td>
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<tr>
<td>$\nu_{\omega_3}=0.1$</td>
<td>4.6806</td>
<td>1.7761e-5</td>
<td>6.3951</td>
<td>0.0046</td>
<td>29.2741</td>
<td>1.4463e6</td>
<td>44.3664</td>
<td>2.2328</td>
</tr>
<tr>
<td>$\nu_{\omega_4}=0.1$</td>
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<td>0.0011</td>
<td>6.3951</td>
<td>3.0496e-5</td>
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<td>1.4738</td>
<td>44.2143</td>
<td>4.3299e-7</td>
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<td>0.9091</td>
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<td>$\nu_{\omega_6}=0.1$</td>
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<td>29.2753</td>
<td>0.2373</td>
<td>44.2143</td>
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<td>4.6805</td>
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<td>0.1336</td>
<td>29.2741</td>
<td>1.9962e-6</td>
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<td>1.2904</td>
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<td>$\nu_{\omega_8}=0.1$</td>
<td>4.6768</td>
<td>0.0394</td>
<td>6.3951</td>
<td>5.5457e-4</td>
<td>29.2400</td>
<td>1.2322</td>
<td>44.2143</td>
<td>1.4359e-7</td>
</tr>
<tr>
<td>$\nu_{\omega_9}=0.1$</td>
<td>4.3879</td>
<td>1.5790</td>
<td>6.4118</td>
<td>0.3002</td>
<td>29.2741</td>
<td>1.3219e-5</td>
<td>44.2162</td>
<td>0.0294</td>
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<tr>
<td>$\nu_{\omega_{10}}=0.1$</td>
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<td>1.7091</td>
<td>6.4089</td>
<td>0.0377</td>
<td>29.2746</td>
<td>0.0091</td>
<td>44.2143</td>
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<tr>
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<td>6.4495</td>
<td>0.4498</td>
<td>29.3639</td>
<td>2.0153</td>
<td>44.3473</td>
<td>2.7629</td>
</tr>
</tbody>
</table>

Table 3. The computational results of random responses

<table>
<thead>
<tr>
<th>Model</th>
<th>$\mu_{v_{z_1}} \times 10^2$</th>
<th>$\sigma_{v_{z_1}}$</th>
<th>$\mu_{v_{z_2}} \times 10^2$</th>
<th>$\sigma_{v_{z_2}}$</th>
<th>$\mu_{v_{z_3}} \times 10^2$</th>
<th>$\sigma_{v_{z_3}}$</th>
<th>$\mu_{v_{z_4}} \times 10^2$</th>
<th>$\sigma_{v_{z_4}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\nu=0$</td>
<td>2.7531</td>
<td>0</td>
<td>61.2658</td>
<td>0</td>
<td>1.0414</td>
<td>0</td>
<td>8.7759</td>
<td>0</td>
</tr>
<tr>
<td>$\nu_{\omega_1}=0.1$</td>
<td>2.7873</td>
<td>73.6973</td>
<td>61.7905</td>
<td>4.9045</td>
<td>1.0245</td>
<td>133.1999</td>
<td>8.7487</td>
<td>112.4368</td>
</tr>
<tr>
<td>$\nu_{\omega_2}=0.1$</td>
<td>2.7309</td>
<td>41.9330</td>
<td>61.8825</td>
<td>2.8877</td>
<td>1.0242</td>
<td>126.1182</td>
<td>8.7191</td>
<td>55.0714</td>
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<tr>
<td>$\nu_{\omega_3}=0.1$</td>
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<td>2.2477</td>
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From Table 2 and 3, it can be obtained that the uncertainty of the vehicle's natural frequencies is dependent on the uncertainty of vehicle's parameters. The randomness of the distances of the rear/front suspension locations to the centre of gravity of the vehicle body, that is geometric parameters $l_2$ and $l_1$, respectively produce the greatest effect on the vehicle’s first and second natural frequency. However, the change of masses of the rear/front wheels, that is $m_{z_2}$ and $m_{z_1}$, respectively produce the greatest effect on the vehicle’s third and fourth natural frequency. The uncertainty geometric parameter $l_1$ and vehicle body’s mass $m_1$ produce greatest effect on the mean square displacement of vehicle body and real wheel, respectively. The randomness of geometric parameter $l_1$ produces notable effect on vehicle’s random responses, especially for rotary angle of the vehicle body and front wheel. Comparing with the case that only one of the uncertainty of vehicle’s parameters is
taken into account, the change of the vehicle’s dynamic characteristics and response are greater when their uncertainty are considered simultaneously.

It should be noted that when the randomness of all vehicle’s parameters are considered, the standard deviations of vehicle’s random response are too big as given in Table 3. Therefore, the probabilistic method seems not applicable and interval analytic methods are more suitable to find the change range (lower and upper bounds) of vehicle’s responses.

4 Conclusions

In this paper, a stochastic half-car model is used to investigate the dynamic response of cars with uncertainty. The effect of uncertainty in the vehicle’s parameters on the randomness of the natural frequencies and vehicle’s random responses are presented by using the MCSM. The dynamic characteristics and random response of stochastic vehicles are obtained expediently. This method will also be applied to the dynamic analysis of random vehicles by using stochastic full-car models.

References