Comparison of added mass coefficients for a floating tanker evaluated by conformal mapping and boundary element methods

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Abstract
One of the important parameters needed to model ship motions in a seaway is the added mass matrix of the hull. Current state-of-the-art boundary element methods routinely evaluate the 6 x 6 added mass matrices as part of the radiation problem solution. These developments have largely superseded conventional approaches to sectional added mass evaluation using conformal mapping techniques. However, conformal mapping techniques are still attractive in terms of their mathematical explicitness and computational simplicity.

The recurrent form of Bieberbach Method of conformal mapping was developed for mapping the exterior of a closed curve i.e. the two-dimensional ship cross section and its mirror image, into the exterior of the circle oscillating vertically at free surface and to compute the added mass coefficients. By incorporating a strip theory approximation the added mass coefficients of a three dimensional structure can be estimated from its two-dimensional section coefficients at different drafts. In this paper we have applied this method to calculate the heave, pitch and heave induced pitch added mass coefficients of a tanker. The applicability of these conformal mapping techniques to floating platforms under consideration is discussed, by comparing the results with state-of-the-art industry standard boundary element methods, AQWA and SESAM.

Introduction
Added Mass is the pressure force per unit acceleration acting on an oscillating floating body, due to the acceleration field set up in the surrounding fluid. It is different in different degrees of motion and depends upon the geometry of the body. The governing equations of motion for a floating rigid body are given by:

\[ (M_0 + a_y) \ddot{x}_y + b_y \dot{x}_y + k_y x_y = F_y(t) \]  

where
\[ M_0 = \text{Oscillating mass/ moment of inertia} \]
\[ a_y = \text{Added mass induced in i due to unit acceleration in j} \]
\[ b_y = \text{Damping} \]
\[ k_y = \text{Restoring stiffness} \]
\[ F_i = \text{Exciting forces in the i}^{th} \text{direction} \]
\[ x_i \dot{x}_i \ddot{x}_i = \text{the displacement, velocity and acceleration of the vessel in the j}^{th} \text{direction} \]
\[ i, j = 1, \ldots, 6, \text{denote the six degrees of freedom surge, sway, heave, roll, pitch and yaw respectively as shown in Figure 1.} \]

An added mass coefficient \( C_{ij} \) is the ratio of the added mass to the mass of the body as given below

\[ C_{ij} = \frac{a_{ij}}{M_0} \]  

Figure 1. 6 DOF of a rigid body

Motion prediction of floating platforms is accomplished using one of the several boundary element packages available in the market. Out of these packages, AQWA [1], and SESAM [2] are used by offshore industry worldwide. The common procedure adopted in these methods is to discretize the underwater surface into discrete elements and solve for the incompressible, irrotational velocity potential of the flow around the body. The boundary value problem is formulated using Green’s function from linear diffraction and radiation theory. One solution outcome is the 6x6 added mass matrix as a function of the frequency of oscillation.

There are no known sources of experimental or 3-D computational results for radiation forces on systematic series of hull forms, which is perhaps not surprising due to the substantial effort involved in either case. On the other hand, such data has been available for 2-D forms for quite some time. Vugts [3] published a comprehensive set of experimental data, along with some theoretical results, for 2-D cylinders including semicircles, triangles, several ship-like sections, and rectangles at range of drafts.

In the ‘60s, Professor Landweber and M. Macagno from Iowa Institute of Hydraulic Research attempted a number of added mass coefficient calculations methods including two parameter, three parameter and conformal mapping developments to calculate added mass coefficients of two dimensional forms [4, 5, 6]. The method of conformal mapping was found comparatively better than the other two methods, and provides higher level of accuracy as compared to the other two methods [7]. The first two methods are based on the assumption that a ship section having the same principal geometric characteristics as a member of one

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of the particular families, the Lewis forms or the three-parameters forms, will have the same added mass coefficient as that member. This assumption is not exact and no level of accuracy can be provided [6]. This method of conformal mapping thus has gone largely unnoticed by software developers.

Once coefficients of a two-dimensional section are known, the coefficients for the entire ship may be found by strip theory (see for e.g. [8] or [9]). This theory is applicable to slender bodies i.e. when the length of the body is much larger compared to the lateral dimension as shown in Figure 2. It is based on the assumption that the radiated wave lengths are of same order of magnitude as beam of the tanker and short as compared to the length of the tanker. Strip theory has the limitation of predicting transverse motions better than longitudinal motions.

We present here the basics of the theory of Landweber and Macagno [6] followed by its implementation into a strip theory formulation. Results for added mass coefficients from this theory are compared with boundary element packages for an offshore floating tanker vessel. It is to be noted that this method is basically applicable to the case of infinite frequency only.

Theory

The radiation problem comprises of a body shape oscillating at an angular frequency $\omega$. The boundary condition on the underwater surface is basically satisfaction of no flow through the surface, and is given in terms of the velocity potential as

$$\omega^2 \phi = g \frac{\partial \phi}{\partial y} \tag{3}$$

The limiting condition when frequency $\omega \rightarrow \infty$ is $\Phi = 0$.

The conformal method is based on mapping the exterior of a strip-section and its mirror image into the exterior of the circle in $\zeta$-plane as shown in Figure 2. Here $B$ and $T$ are the beam and draft of the floating body respectively. Using a double-body contour eliminates free surface influence in the problem. The boundary condition is satisfied by supposing that the entire shape oscillates as a single shape with instantaneous velocity $V$ which gives

$$\frac{\partial \phi}{\partial n} = V \frac{\partial y}{\partial n} \tag{4}$$

The above equation relates the velocity potential to the shape of the contour. At infinite frequency, the amplitude of oscillation diminishes infinitesimally, resulting in a finite velocity. The kinetic energy resident in the fluid due to this oscillation may be directly related to the added mass as

$$KE = \frac{1}{2} a_{33} V^2 = -\rho \int \phi \psi \ d\mathcal{V} \tag{5}$$

where $\psi$ is the stream function. Solving for KE subject to the boundary conditions gives the added mass coefficient as

$$C_{33} = C_v = \frac{1}{b^3} \left[ 2 \left( v_o^2 + b_1 \right) - \frac{S}{\pi} \right] \tag{6}$$

Where

- $C_v$ = Added Mass Coefficient due to vertical (heave) oscillations
- $b$ = half beam of the ship section
- $r_o$ = Mean radius of the ship section
- $S$ = Area of the ship section
- $b_1$ = the first coefficient of mapping from $z$-plane to $\zeta$-plane.

It remains to find the parameters $b_1, r_o$ and $S$. Let us consider a double ship like contour $cc'$, symmetrical with respect to the $x, y$ axes with the $x$-axis in the free surface as shown in Figure 3. The transformation as per the Bieberbach method of conformal mapping is given by [6]

$$z = \zeta + \frac{a_1}{z^3} + \frac{a_2}{z^5} + \frac{a_3}{z^7} + \frac{a_4}{z^9} + \cdots \tag{7}$$

Where $z = re^{i\theta}$ is the complex coordinate. Its inverse given by

$$\zeta = z + \frac{b_1}{z} + \frac{b_2}{z^3} + \frac{b_3}{z^5} + \frac{b_4}{z^7} + \cdots \tag{8}$$

is the transformation of mapping the exterior of the closed curve (Ship section and its mirror image) in the $z$-plane into the exterior of the closed circle in $\zeta$-plane (Figure 3). The form of these mapping functions as expansions of odd powers of $z$ and $\zeta$ with real coefficients is required in order to satisfy the condition that both the original section $cc'$ and its transformation in $\zeta$-plane be symmetrical with respect to their coordinate axes. Here $a_i$ and $b_i$ are the coefficients of the mapping.
The Bieberbach method is based on the property that among the closed curves obtained from the conformal mapping, by transformations of Eq. 8, of the exterior of the given closed curve bounding a simply connected region, the circle will bound the maximum area. Ritz procedure is applied to find the values of finite number of coefficients of mapping i.e. \( b_j \)'s in Eq. 8 such that they yield maximum area subject to this restriction.

The area bounded by the curve CC' as shown in Figure 3 is given by

\[
S = \frac{1}{2} \oint_{c'} z' dz = i \oint_{c'} r^2 d\theta = 2iS
\]  
\[ (9) \]

Thus we get the area as

\[
S = \frac{1}{2} \oint_{c'} r^2 d\theta
\]
\[ (10) \]

Let \( C_1 \) be the closed curve in \( \zeta \)-plane obtained by mapping CC' one-to-one in \( z \)-plane. From Eq. 8, we obtain

\[
\oint_{C_1} \bar{z}' d\zeta = 2iS
\]
\[ (11) \]

Substituting Eq. 8, we get

\[
\oint_{C_1} \bar{z}' d\zeta = \oint_{C'} \left[ -\sum_{j=1}^{n} \frac{b_j}{z} \right] \left[ 1 - \sum_{k=1}^{n} \frac{(2k-1)b_k}{z^{2k-1}z^2} \right] dz
\]
\[ (12) \]

Substituting values in Eq. 12 from Eqs. 9 and 11, the area, \( S' \) bounded by this curve is given by,

\[
S' = S - \sum_{j=1}^{n} \sum_{k=1}^{n} \beta_{jk} b_j b_k + 2 \sum_{j=1}^{n} e_j b_j
\]
\[ (13) \]

Where

\[
e_j = -\frac{i}{4} \oint_{c'} \left[ \frac{1}{z} - \frac{2j-1}{z^2} \right] dz
\]
\[ (14) \]

\[
\Rightarrow e_j = \begin{cases} \frac{2j-1}{2j-2} \int_0^{2\pi} \cos 2j\theta d\theta, & j > 1 \\ -\int_0^{2\pi} \ln r \cos 2\theta d\theta, & j = 1 \end{cases}
\]

And

\[
\beta_{jk} = -\frac{i}{4} \oint_{c'} \left[ \frac{2j-1}{z} - \frac{2k-1}{z^2} \right] dz
\]
\[ (15) \]

\[
\Rightarrow \beta_{jk} = \frac{(2j-1)(2k-1)}{2(j+k-1)} \int_0^{2\pi} \cos(2j-2k)\theta d\theta
\]

Simpson’s Rule of integration may be used for the integration of Eq.15. For Ritz condition that \( S' \) be a maximum Area, the derivative of \( S' \) with \( b_j \) should be zero.

\[
\frac{\partial S'}{\partial b_j} = 0, j = 1, 2, ..., n
\]
\[ (16) \]

Using Eq. 13 we get a set of linear equations with the coefficients \( b_k \),

\[
\sum_{k=1}^{n} \beta_{jk} b_k = e_j
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This system of linear equation is solved to get \( b \)'s, the values of coefficients of mapping.

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The data from conformal mapping shows approximately constant value for various \( B/2T \) with a slight decrease in the limit of \( B/2T \) → 0 i.e. a very deep structure. In contrary Vugts data shows a very small value of \( a_{33} \) as \( B/2T \)→0. The agreement is seen to be
reasonable at realistic values of $B/2T$ of around 2 (as shown in Figure 4). At very high values of $B/2T$ (shallow draft) the conformal mapping shows an increase in the added mass coefficient, which is contrary to Vugts’s data. The shallow draft limitation of strip theory is well known (for example, Lewis [9]) and is being studied further. Landweber and Macagno [4] mentioned that the mapping twice gives more accuracy in the added mass coefficient values. This aspect will be considered in the future work.

Figures 5 and 6 show curves of heave-induced pitch ($A_{15}$) pitch-induced pitch added mass ($A_{25}$) and pitch induced heave ($A_{15}$) vs. $B/2T$. Since these are obtained from Eqs. 17 and 18, they follow an amplified monotonic trend derived from Figure 4.

The added mass values and corresponding coefficients of the entire hull were independently computed using both AQWA and SESAM for two different drafts. As is well known, computations with boundary element software becomes quite unreliable at very low wave lengths, corresponding to higher frequencies. For realistic offshore structures this limits application of these methods to periods less than 5 seconds (frequencies of 0.2 Hz). The computational results at the highest frequency are compared with the present strip theory results in Tables 2 and 3. The agreement is shown to be very reasonable with maximum errors of less than 10% in all cases.

Conclusions

We have presented the theory and implementation of a conformal mapping technique and compared with published data as well as with industry standard software results. Following conclusions have been drawn.

- Conformal mapping method of calculating added mass coefficients is quicker for calculations at varying drafts as compared to the other two industry accepted methods.
- It achieves the level of accuracy in calculating added mass of the three dimensional floating tanker.

Further studies in progress:

- Analysis of the increase in the 2-D sectional added mass coefficient at higher $B/2T$, which is contrary to the Vugts data.
- Repetitive application of conformal mapping to achieve higher level of accuracy

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References