Experimental investigation of quantum key distribution with position and momentum of photon pairs

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We investigate the utility of Einstein-Podolsky-Rosen correlations of the position and momentum of photon pairs from parametric down-conversion in the implementation of a secure quantum key distribution protocol. We show that security is guaranteed by the entanglement between down-converted pairs, and can be checked by either direct comparison of Alice and Bob’s measurement results or evaluation of an inequality of the sort proposed by Mancini et al. [Phys. Rev. Lett. 88, 120401 (2002)].

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I. INTRODUCTION

Quantum-communication protocols using photonic qubits have been proposed and implemented, utilizing entanglement in several degrees of freedom of the photon [1]. Perhaps the most promising application of quantum communication using photons is quantum key distribution (QKD) [1,2], in which entangled or single qubits are sent from sender to receiver(s) and used to establish a secret random key string, which can then be used to securely transmit a data string. Most QKD schemes are inspired by the original single-qubit Bennett-Brassard 1984 (BB84) [3] or entangled-qubit Ekert [4] protocols. In the BB84 protocol, cryptographic security is provided by the partial indistinguishability of non orthogonal states and the no-cloning theorem, while the security of the Ekert protocol is guaranteed by violation of Bell’s inequality.

For photon pairs, obtained from spontaneous parametric down-conversion (SPDC), entanglement in polarization [5] and energy-time [6] have been most widely and successfully used. There have also been QKD proposals based on continuous variable-field quadratures of multiphoton beams [7]. Again, security of these protocols is based on either measurement of non commuting observables and the no-cloning theorem or violation of some classical inequality.

In this paper, we present an experiment which demonstrates a protocol for QKD based on the position and momentum degrees of freedom of entangled photons created by SPDC. Recently, it has been shown that the difference between the positions of entangled SPDC photons along with the sum of their momenta are Einstein-Podolsky-Rosen (EPR) correlated, or, in other words, entangled [8]. These correlations are interesting not only for their direct relation to the seminal paper by EPR [9], but also for their possible application to quantum-information tasks. However, it is not obvious that position and momentum degrees of freedom can be used to represent a qubit in the same way that is done with polarization. In this regard, it is not clear as to what extent the position and momentum degrees of freedom of photon pairs can be used in quantum-communication protocols such as, for example, quantum cryptography, quantum teleportation, etc. Here we provide an experimental investigation that shows that the position and momentum degrees of freedom of photon pairs can be used to implement a secure quantum key distribution protocol. To our knowledge, the work we present here is the first quantum-information protocol based on position and momentum entanglement of the form originally proposed by EPR.

Any practical QKD scheme should be viable over long distances. Under ideal conditions, long-distance implementation of our QKD scheme presents no problem, since it has been demonstrated that it is possible to protect entanglement of transverse spatial degrees of freedom from divergence due to free-space (noise-free) propagation of the down-converted photons [11]. As in any implementation of a free-space quantum communication system, however, free-space transmission can cause errors due to air turbulence, absorption, scattering, etc. Absorption and scattering lead to losses, which of course reduces the data transmission rate. But as long as the losses are below a given threshold, they do not compromise the security of the QKD protocol [2]. Errors caused by fluctuations due to air turbulence and other factors which do not lead to losses can deteriorate the correlation between the photon pairs, which leads to an increased bit error rate. However, these fluctuations can be accounted for using a classical reference beam, as in [12]. We will further discuss long-distance implementation in Sec. V.

Our paper is organized as follows. In Sec. II we present the QKD protocol, and discuss basic security issues in Sec. III. Section IV shows experimental results confirming the utility of position and momentum EPR correlations in QKD. In Sec. V we discuss potential advantages provided by our system as well as the implementation of our protocol over long distances.

II. THEORY

Figure 1 shows a diagram of the basic QKD apparatus. The protocol works as follows. Twin photons are created by a SPDC source. One photon of each pair is sent to Alice (A), and one to Bob (B). At each of Alice and Bob’s stations, a random selection between detection in the position (x) or...
FIG. 1. (Color online) Diagram of quantum key distribution using position and momentum variables of down-converted photons. The SPDC source emits photons entangled in position and momentum. Lenses are used to create the imaging (position measurements) or Fourier transform (momentum measurements) in Alice and Bob’s detection planes. The 50-50 non-polarizing beam splitters (BS) are used to choose randomly between position and momentum measurements.

momentum ($p$) basis is performed by a 50-50 non-polarizing beam splitter. A measurement in the $x$ basis is easily performed by a lens imaging the crystal onto the detection plane. It is well known that a measurement in the $p$ basis can be implemented by a lens with crystal and detection planes in its focal planes [13], as was done in Refs. [8].

Before any photons transmitting potential key bits are exchanged, Alice and Bob calibrate their $x$ detectors for positions $Ax_1$ and $Ax_2$ and $Bx_1$ and $Bx_2$, and their $p$ detectors at positions $Ap_1$ and $Ap_2$ and $Bp_1$ and $Bp_2$, respectively. Let us assume that, in both bases, detector position “1” (“2”) represents the 0 (1) logical level. From the multimode theory of SPDC [11,14], it is possible to show that, for ideal point like detectors, the probability to detect photon pairs is

$$P_{xx}(p_A,p_B) \propto |\mathcal{W}(\alpha p_A + \beta p_B)\delta(\alpha p_A - \beta p_B)|^2 \quad (1)$$

for a position-position ($xx$) configuration. The notation “ij” refers to the situation where Alice measures in the $i$ basis and Bob in the $j$ basis, where $i,j=x,p$. Here $\mathcal{W}$ is the field profile of the pump beam at the crystal face. We define the parameters $\alpha = O_A/(2iA)$ and $\beta = O_B/(2iB)$, which are related to the magnification factors of Alice’s and Bob’s imaging systems. We have also assumed that both imaging lenses have been placed so as to obey the thin-lens equation. The $\delta$ function appears due to the fact that we have considered ideal point like detectors. Since the pump beam profile $\mathcal{W}$ can be made much larger than the $\delta$ function, it is possible to find positions $p_A$ and $p_B$ such that $\mathcal{W}$ is approximately constant.

Similarly, for a momentum-momentum ($pp$) configuration,

$$P_{pp}(p_A,p_B) \propto \left| v \left( \frac{k_A}{f} p_A + \frac{k_B}{f} p_B \right) \right|^2 \quad , \quad (2)$$

where $v$ is the angular spectrum of the pump beam at the crystal face. If they measure in the same basis, Alice and Bob can calibrate their detectors so that they see a large correlation in their coincidence measurements. For example, if Alice detects a photon at $Ax_1$, then the probability for Bob to detect a photon at $Bx_1$ will be much greater than the probability to detect a photon at $Bx_2$. In general, Alice and Bob can choose positions such that

$$P(Ax_1,Bx_1) = P(Ax_2,Bx_2) \gg 0, \quad (3)$$

$$P(Ax_1,Bx_2) = P(Ax_2,Bx_1) = 0, \quad (4)$$

where $P(A,B)$ is the coincidence detection probability. The same is true for measurements in the $p$ basis:

$$P(Ap_1,Bp_1) = P(Ap_2,Bp_2) \gg 0, \quad (5)$$

$$P(Ap_1,Bp_2) = P(Ap_2,Bp_1) = 0. \quad (6)$$

Moreover, Alice and Bob can adjust their detection systems such that $P(Ax_1,Bx_1) = P(Ax_2,Bx_2) = P(Ap_1,Bp_1) = P(Ap_2,Bp_2)$. In the interest of simplicity, we will remove the subscripts on the probabilities $P_{ij}$ whenever redundant.

The security of the QKD protocol is conditioned on the fact that the correlation is low when the measurements are performed in different bases, so that very little information is shared. For example, if Alice detects a photon at $Ax_1$, then the probability for Bob to detect a photon at $Bp_1$ should be the same as the probability to detect a photon at $Bp_2$. Using the standard theory of SPDC, the coincidence detection probability for an $xp$ configuration is

$$P_{xp}(p_A,p_B) \propto |\mathcal{W}(\alpha p_A)|^2 \quad , \quad (7)$$

while a $px$ configuration gives

$$P_{px}(p_A,p_B) \propto |\mathcal{W}(\beta p_B)|^2 \quad , \quad (8)$$

There is no correlation in the detection probabilities (7) and (8). Thus, Alice and Bob can choose their detector positions such that $P(Ax_1,Bp_1) = P(Ax_1,Bp_2) = P(Ax_2,Bp_1) = P(Ax_2,Bp_2) = P(Ap_1,Bx_1) = P(Ap_1,Bx_2) = P(Ap_2,Bx_1) = P(Ap_2,Bx_2)$.

To generate a secret key, Alice and Bob perform a series of measurements on a number of photon pairs until they have accumulated $N$ coincidence events, where $N$ depends on the size of the key and the level of security required. All events in which only one or neither of them detect a photon are discarded. After the $N$ photon pairs have been detected, Bob, through classical communication, informs Alice of his measurement basis ($x$ or $p$) for each photon. On average, Alice and Bob measure in the same basis $50\%$ of the time. They keep these photons and discard the rest. Alice then chooses $m$...
III. SECURITY

We will now discuss the security of this protocol. There has been much work in security proofs for a wide variety of attacks on QKD systems [2]. We will limit our discussion to incoherent attacks, in which the eavesdropper (Eve) has access to one photon pair at a time. Here we provide a security argument for a simple attack, and leave more complex attacks for future work.

If Eve steals one or both photons, then no coincidence is detected, the event is discarded and no information is obtained by Eve. One way for Eve to obtain information is based on an intercept-resend type strategy [2], in which she steals one or both photons, measures (and thus destroys) them, and then replaces them with new photons. We note here that we call this type of general attack, in which Eve can measure in any basis, an intercept-resend “type” strategy, in contrast to the usual intercept-resend strategy found in the literature in which Eve measures in Alice and Bob’s bases. As in the BB84 or Ekert protocols using qubits, in this attack Eve’s presence is marked by a deterioration of the correlation in Alice and Bob’s measurements.

One method of testing security is through the quantum bit error rate (QBER), which can be defined as a function of the “wrong” and “right” detection probabilities [2]:

$$(\text{QBER}) = \frac{P_{\text{wrong}}}{P_{\text{wrong}} + P_{\text{right}}}.$$  \hspace{1cm} (9)

In order to simplify the analysis of our experimental results, we will not assume $P_{\text{wrong}} + P_{\text{right}} = 1$. With no eavesdropper present, in our notation the QBER is given by

$$\text{(QBER)}_0 = \sum_{j=x,p} \sum_{x,1,2} P(A_{j,1},B_{j,1}).$$

(10)

There are many different attack strategies that Eve can adopt [2]. As an example, let us consider that Eve implements the usual intercept-resend strategy on Bob’s quantum channel. Let us suppose that Eve is completely aware of Bob’s detection system and has constructed an identical one of her own, and denote $R_{ij}(\rho_A,\rho_B)$ as the coincidence detection probability for Alice and Eve, where $i,j=x, p$. Then, Alice and Eve’s detection probability is the same as that of Alice and Bob: $R_{ij}(\rho_A,\rho_B) = R_{ji}(\rho_A,\rho_B)$. She intercepts and measures Bob’s photon, giving one of the following results: $E_{x}, E_{x}^2, E_{p}, E_{p}^2$, or a null count. She then prepares a photon in the eigenstate corresponding to her measurement result and sends it to Bob. We will consider only the cases where Eve detects a photon. Then, for a given photon pair, Alice, Bob, and Eve detect photons with probability

$$P_{ij}(\rho_A,\rho_B,\rho_E) = R_{ij}(\rho_A,\rho_E)p(\rho_B;k),$$

(11)

where $i,j,k=x$ or $p, p(\rho_B;k)$ is the probability that Bob will detect Eve’s replacement photon in basis $j$ given that it was prepared in basis $k$, and we now limit ourselves to the cases where $\rho_A, \rho_B$, and $\rho_E$ are one of Alice, Bob, or Eve’s pre-defined measurement positions $(A_{x,1}, B_{x,1}, E_{x,1}, \ldots)$. If Eve chooses the same measurement basis as both Alice and Bob, then she can go essentially undetected, since

$$P_{ii}(\rho_A,\rho_B,\rho_E) = R_{ii}(\rho_A,\rho_E) = P_{ii}(\rho_A,\rho_B),$$

(12)

which is the detection probability that Alice and Bob expect. Here we have assumed Eve’s best-case scenario, in which, given that she is completely aware of his detection system, she can replace Bob’s photon in such a way that $p(\rho_B;i) = 1$. However, if Eve chooses the wrong basis, then, for the cases in which Alice and Bob expect a larger detection probability $P_{ij}(\rho_A,\rho_B)$:

$$P_{ij}(\rho_A,\rho_B,\rho_E) = R_{ij}(\rho_A,\rho_E)p(\rho_B;i) < P_{ii}(\rho_A,\rho_B).$$

(13)

when $i \neq j$. This follows from the fact that $R_{ij}(\rho_A,\rho_E) = P_{ij}(\rho_A,\rho_E) < P_{ii}(\rho_A,\rho_B)$ and $p(\rho_B;j) < 1$. In other words, the eavesdropping reduces the correlation between Alice and Bob’s measurements. Similarly, if Eve intercepts the pair of photons by another equally entangled pair, she is essentially playing the role of the source and the protocol is not affected [16]. If she replaces the pair of photons by non- or less-entangled photons, then similarly the correlation between Alice and Bob’s measurements is reduced and Eve’s presence can be detected.

Assuming that Eve measures every one of Bob’s photons in a basis $(x, p)$ chosen at random, the QBER for Eve’s intercept-resend strategy is

$$\text{(QBER)} = \sum_{j=x,p} \sum_{x,1,2} P(A_{j,1},B_{j,1}) + \chi,$$ \hspace{1cm} $\sum_{j=x,p} \sum_{x,1,2} P(A_{j,1},B_{j,1})$ \hspace{1cm} (14)

where

$$\chi = \sum_{j=x,p} \sum_{x,1,2} \sum_{k} p_{ij}(B_{j,1};k)P(A_{j,1},B_{j,1})$$

(15)

is the error due to Eve’s disturbance when she measures in the wrong basis. Here we have assumed that $R(A_{j,1},E_{j,1}) = P(A_{j,1},B_{j,1})$. 

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From the argument above it is demonstrated that eavesdropping reduces the correlation between Alice and Bob’s measurements, and that this correlation is guaranteed by an EPR-like state. The EPR character can be demonstrated, as in Ref. [8], by satisfying the inequality

\[ \Delta^2(Ax_1 - Bx_1)\Delta^2(Ap_j + Bp_j) \leq \frac{\hbar^2}{4}. \]  

To check for the presence of Eve, Alice and Bob can either check the correlations between \( m \) pairs of their \( N \) measurement results by calculating the QBER, or use these \( m \) pairs to verify Eq. (16).

IV. EXPERIMENT

We have performed measurements which demonstrate the security of a QKD protocol using position and momentum variables by testing the correlations between position-position and momentum-momentum coincidence detections, as well the non-correlation between position-momentum and momentum-position detections in a typical twin photon setup. The experimental configuration is shown in Fig. 2. We use a femtosecond pulsed Ti-sapphire laser doubled by a 2-mm-long barium bismathate crystal, obtaining a violet beam with wavelength centered around 425 nm. Using the violet beam to pump a 1-cm-long lithium iodate crystal, down-converted signal and idler photons were produced and detected in different wavelengths with interference filters centered at 890 nm with 10 nm bandwidth and 810 nm with 50 nm bandwidth. We use avalanche photodiode single-photon-counting modules equipped with short-focal-length lenses (often called objective lenses, because the focal lengths are short, but they play the role of oculars) and a thin slit at the entrance. The slits are oriented so that the horizontal dimension is 3 mm and the vertical dimensions are 0.2 mm for position measurements and 0.5 mm for momentum measurements. When detection is performed in the position basis, the crystal face is imaged by a 15-cm-focal-length lens placed 20 cm from the crystal. The detectors are placed 60 cm from the lenses, giving a magnification factor of 3, which allows us to image a narrow region with the 0.2 mm detection slits. The pump beam is a Gaussian beam with spot size \( \approx 2 \) mm at the crystal face. For measurements in the momentum basis, a 15-cm-focal-length lens is used and the detectors are placed 30 cm from the crystal face, so that the crystal plane and the detection plane coincide with the focal planes of the lens.

Results of the experimental investigations are displayed in Figs. 3–6. All curves are Gaussian fits and error bars represent errors due to Poissonian photon-counting statistics. In all measurements detector \( A \) was kept fixed while \( B \) was scanned linearly in the vertical direction. Setting detectors \( A \) and \( B \) for position measurements, we expect to see a large correlation in coincidence measurements. Figure 3 shows the coincidence count rate when detector \( B \) is scanned along the vertical axis and detector \( A \) is fixed at position \( Ax_1 \) (triangles) and also when \( A \) is fixed at \( Ax_2 \) (circles). The peaks of the coincidence distributions are separated by about 1 mm, which is also the separation between \( Ax_1 \) and \( Ax_2 \), implying a good position correlation. The width of these distributions is basically defined by the convolution of the slit apertures in both detectors [14].

Figure 4 shows the coincidence count rate when momentum measurements are performed at both stations. Detector \( B \)
The security of our QKD protocol is based on a large correlation when measurements are performed in the same basis and low correlations when measurements are performed in complementary bases. The measurements in Figs. 3 and 4 show that the correlation between position-position measurements and momentum-momentum measurements are very high. Now we are going to demonstrate that the correlation between position-momentum and momentum-position is negligible.

The coincidence profile for the \( xp \) configuration is shown in Fig. 5. Here detector \( B \) is scanned in the \( p \) configuration with detector \( A \) fixed at \( A_{x1} \) (triangles) and \( A_{x2} \) (circles). There is no correlation between Alice and Bob’s measurements in this case since the coincidence profile is approximately constant for all positions of detector \( B \), as expected from Eq. (7). The slight “enveloping” visible in the coincidence counts is due to a small modulation in the single counts as detector \( B \) is scanned (not shown). Detector \( A \) positions \( A_{x1} \) and \( A_{x2} \) were chosen so that the coincidence rate is approximately the same for both cases. We will show below that this is necessary to guarantee the security of the distributed key.

Figure 6 shows the coincidence profile for the \( px \) configuration, where \( B \) is scanned in the \( x \) basis with detector \( A \) fixed at \( A_{x1} \) (triangles) and \( A_{x2} \) (circles). As expected from Eq. (8), the coincidence count rate reproduces the transverse intensity profile of the pump field as a function of Bob’s detector position only, which shows that in this case there is no correlation between Alice and Bob’s measurements.

Using the results in Figs. 3–6, Bob can choose his detector positions so that the security conditions discussed in Sec. II are best satisfied. Examining the figures, it is most advantageous to define \( B_{x1}=1 \) mm, and \( B_{x2}=2 \) mm, \( B_{p1}=1 \) mm and \( B_{p2}=2 \) mm. Table I shows the coincidence counts for every possible measurement at Alice and Bob’s stations, from which it can be seen that the correlation between measurements in the same basis and the non correlation between measurements in different bases. There is a slight discrepancy among the on-diagonal coincidence count rates. A fine-tuning of the coincidence levels can always be achieved by using neutral filters in front of the detectors so that the on-diagonal terms are approximately equal.

From Table I and Eq. (10), the QBER in the absence of an eavesdropper would be \( \text{QBER}_0 = 0.047 \pm 0.001 \). The QBER for \( xx \) measurement is \( \text{QBER}_{0x} = 0.064 \pm 0.001 \) and \( \text{QBER}_{0p} = 0.027 \pm 0.001 \) for \( pp \) measurements. The security criterion for most QKD protocols is a QBER of less than about 0.15 [2]. For a QBER above this limit it is impossible.

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<td>( B_{p2} )</td>
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Table I. Coincidence counts for Alice and Bob’s chosen detector positions.
to establish a secret key, even with one-way error correction and privacy amplification. Since our QBER is much lower than this limit, it should be possible for Alice and Bob to establish a secure secret key.

We can use the $xp$ and $px$ measurement results to predict the effect of an eavesdropper. Using the results in Table I in Eq. (14), the estimated QBER if Eve were to measure every one of Bob’s photons in a randomly chosen basis ($x$ or $p$) would be $\text{QBER} = 0.296 \pm 0.001$ where we have assumed that $p_j(B_j,k) = 1/2$ for all $r$ and $j \neq k$ in Eq. (15). In other words, we have assumed that if Eve measures in the wrong basis, she has a 50% chance of sending the “correct” replacement photon. We see that Eve’s presence is clearly marked by an increase in the QBER.

V. DISCUSSION

We will first mention two possible advantages that can be exploited in our QKD system and then describe possible implementation over long distances in free space.

(a) Possible advantages. One possible advantage provided by the use of these degrees of freedom is that entanglement in position and momentum is easily obtained, since it is a direct consequence of the inherent phase matching conditions in the parametric process, and can be obtained using thick non linear crystals, which allows for higher-flux sources. Another possible advantage is that position and momentum are continuous variables, which opens the possibility of using higher-dimensional alphabets. For instance, it is possible to use $D$-dimensional spatial qudits as described in Ref. [10] in an improved protocol which would allow for a higher data transmission rate.

(b) Long-distance implementation. As in all experiments involving free-space transmission of entangled photons over long distances, there are two main problems. (i) Losses by absorption or scattering: This kind of problem is independent of the degree of freedom used. Therefore, for the same wavelength and beam diameter, the losses are the same for polarization, time bin, position-momentum, or any other degree of freedom of the electromagnetic field. (ii) Errors which result in a rotation in phase space: In free-space propagation, these errors could possibly be caused by turbulence or fluctuations of the refractive index of the atmosphere. It is reasonable to suppose that free-space propagation is a unitary evolution, since (a) losses will be filtered out by coincidence detections and (b) the coherence time (on the order of hundreds of femtoseconds) of the down-converted photons is so short that it would be unrealistic to consider that fluctuations of the refractive index of the atmosphere occurred at a similar time scale. Therefore, any error results in a rotation in phase space. However, the correlation between photon pairs is maintained, since local unitary operations cannot destroy entanglement. The problem is that we no longer know in which basis to measure in order to observe the appropriate correlations. This problem can be overcome by use of a reference beam, such as used in other schemes [12]. The classical reference beam is setup in such a way as to experience the same propagation as the down-converted fields. In practice, the reference beam would be dynamically monitored in such a way as to make dynamical corrections to the detection systems, i.e., adjustments to the positions of Alice and Bob’s imaging and Fourier transform lenses. We note that changing the lens position is equivalent to performing a rotation in phase space. In this way, errors can be detected and minimized.

VI. CONCLUSION

Recent work [8] has shown that photon pairs created by spontaneous parametric down-conversion exhibit entanglement in position and momentum, of the sort originally proposed in the fundamental paper by Einstein, Podolsky, and Rosen [9]. Here we have extended these ideas by experimentally investigating the implementation of a quantum key distribution protocol based on the quantum correlations between position and momentum of entangled photon pairs. In addition to interest due to the relation with historical debates on quantum theory, quantum key distribution based on position and momentum of photon pairs might offer some advantages. First, using these degrees of freedom allows for the use of long nonlinear crystals, therefore opening the possibility of having really high flux entangled-photon sources. Second, it might be possible to extend these results to higher-dimensional systems as in Ref. [10], which could be used in quantum-communication protocols such as quantum bit commitment [18].

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