Efficient Frequency Estimation
and Time-Frequency Representations

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Abstract: A computationally efficient estimator of the frequency of a single complex sinusoid in complex white Gaussian noise was recently proposed by Kay. Although it is much simpler than the optimal maximum likelihood (ML) estimator, Kay claims that it can achieve the same statistical performance at moderate signal-to-noise ratios (SNRs). In fact, we show that this estimator is biased and has a frequency dependent threshold which is often much greater than the 6 dB claimed. Kay's estimator is related to certain members of the class of smoothed central finite difference (SCFD) instantaneous frequency (IF) estimators which were developed while investigating frequency estimation via discrete-time frequency representations (TFRs) in Cohen's class. Combining Kay's analysis with our own knowledge, we now introduce the parabolic SCFD (PSCFD) IF estimator which is unbiased, optimal and has a frequency independent threshold of 8 dB.

1 Introduction

In [1] and [2] the class of smoothed central finite difference (SCFD) instantaneous frequency (IF) estimators was introduced. This class of estimators is closely related to discrete time-frequency representation (TFR) moment IF estimators obtained via the periodic first moment [1] of TFRs with respect to frequency. In response to Kay's recently proposed estimator [3], we now introduce the parabolic SCFD (PSCFD) estimator which is optimally efficient since it meets the Cramer-Rao lower bound for moderate signal-to-noise ratios (SNRs).

2 Discrete-Time Definitions

First we give some definitions and conditions from [1].

Definition 1 Discrete-Time Analytic Signal: The discrete-time analytic signal \( z \) associated with the real discrete-time signal \( x \) is defined by

\[
\begin{align*}
    z &= A[x] \\
    &= x + jH[x]
\end{align*}
\]

where \( A[\cdot] \) is the linear operator which forms the analytic signal and \( H[\cdot] \) is the discrete-time Hilbert transform defined by

\[
H[x](n) = \sum_{m \text{ odd}}^{+\infty} \frac{2x(n - m)}{m\pi}.
\]

We use the central finite difference (CFD) of the phase of \( z \) as the analogue of the continuous-time Analytic Derivative IF estimator.

Definition 2 CFD IF Estimator: Let \( z = A[x] \) where \( x \) is a real discrete-time signal. Then the IF of \( x \) at sample \( n \) is estimated by

\[
\hat{f}_D(n) = \frac{1}{4\pi} \left( \arg[x(n + 1)] - \arg[x(n - 1)] \right) \quad (3)
\]

where \( (\cdot)_{2\pi} \) represents reduction modulo \( 2\pi \) and \( f_s \) is the sampling frequency.

Although Cohen's [4] class of TFRs is usually formulated in the doppler-lag domain, it is convenient to reformulate it in the discrete-time time-lag domain as follows:

Definition 3 Cohen's Class of TFRs for Discrete-Time Analytic Signals: Each member of this class of bilinear TFRs can be written in the form:

\[
S(n, k) = \sum_{m=-L}^{+L} \sum_{p=-L}^{+L} B(p-n, m)x(p+m)x^*(p-m)e^{-j2\pi km/N}
\]

\[
= F_{m-k} \left[ B(-n, m) \ast C(n, m) \right] (n)
\]

where \( N = 2L + 1 \) is the length of the real signal \( x \), \( F_{m-k} \) represents discrete Fourier transformation from \( m \) (lag) to \( k \) (frequency), \( B(n, m) \) is the discrete-time time-lag kernel function which characterizes the TFR, \( C(n, m) = x(n + m)x^*(n - m) \) is called the bilinear product, and \( \ast \) represents linear convolution in the \( n \) index.

The quantity \( B(-n, m) \ast C(n, m) \) is just a generalization of the autocovariance estimate of \( x \). Linear convolution is inappropriate for circular quantities such as discrete-time IF estimators. In such cases the modulo-\( \lambda \) convolution operation from [1] should be used.

Definition 4 Modulo-\( \lambda \) Convolution: Let the sequence \( f(k) \) of the form \( f(n) = ((f(n)), \lambda, f : Z \rightarrow \mathbb{R} \) and \( \lambda \in \mathbb{R} \). If we convolve \( f \) with a smoothing function \( h \) of odd length \( P = 2Q + 1 \), \( h : Z \rightarrow \mathbb{R} \), then we must use the modulo-\( \lambda \) convolution operation defined by

\[
\hat{f}(n) ((\cdot)_{2\pi}) \lambda(h) = \frac{\lambda}{2\pi} \left( \arg \left[ \sum_{p=-Q}^{Q} h(p)e^{2\pi j (n-p)/\lambda} \right] \right) \quad (5)
\]

The behaviour of TFR moment IF estimators is related to the class of smoothed CFD IF estimators defined as follows:

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Definition 5 SCFD IF Estimator: Let \( \hat{f}_n^2 \) be the CFD IF estimator calculated from the real signal \( x \) and let \( h : Z \rightarrow R \), be a smoothing function of odd length \( P \). Then the SCFD IF estimator is defined by

\[
\hat{f}_n^2 = B(-n, 1) \hat{f}_{n/2} \hat{f}_{n/2}, \quad \forall n.
\]

(6)

If we have a monocomponent, constant amplitude FM signal then the TFR moment IF estimator is given by

\[
\hat{f}_n = \frac{B(-n, 1) \hat{f}_{n/2}}{\hat{f}_{n/2}^2}, \quad \forall n.
\]

(7)

and so, in this case, the periodic first moment \( f_1 \) of a discrete-time TFR is identical to the SCFD estimator. The smoothing function used to calculate the estimator is given by the discrete-time time-lag kernel cross-section at \( m = 1 \).

In the case of a tone in noise, the SCFD estimator will always have a variance which is slightly less than the corresponding TFR estimator due to variations in \( \hat{f}_{n+1}^2 - \hat{f}_{n-1}^2 \).

The following theorem comes from the analysis in [5]. It applies to a noisy signal \( z \) comprising a monocomponent FM tone with amplitude \( \sigma_c \) and IF law \( f_t \) in white Gaussian noise of variance \( \sigma_n^2 \).

Theorem 1 Distribution of \( \hat{f}_n^2 \): Let \( \hat{f}_n^2 \) be the SCFD IF estimator calculated from the CFD estimaor \( f_t \), using the modulo convolution operator with a non-negative smoothing function \( h \) of odd length \( P = 2Q + 1 \). Then \( f_t(n) \) is distributed approximately as wrapped normal \( N(f_t(n), \sigma_c^2) \) and \( \hat{f}_n = \frac{B(-n, 1) \hat{f}_{n/2}}{\hat{f}_{n/2}^2} \) is distributed approximately as \( N(f_t(n), \sigma_c^2) \) where

\[
\hat{f}_n = f_t(n) \frac{(\ast)}{\hat{f}_{n/2}} h(n)
\]

(8)

and

\[
\sigma_c^2 = \frac{\tilde{\eta}^2 \phi_h}{(h^2 1^2)^2} \hat{f}_n^2.
\]

(9)

and \( \sigma_c \) is the dispersion of \( \hat{f}_n^2 \) given by \( \sigma_c = f_s/4 \pi \sqrt{s} \) where \( s = \sigma_n^2/2 \sigma_c^2 \) is the SNR. The \( P \) dimensional smoothing function vector and unit vector are defined by

\[
h = [h_0, \ldots, h_{-Q}]^T
\]

(10)

and

\[
1 = [1, \ldots, 1]^T
\]

(11)

respectively; the \( P \times P \) dimensional correlation covariance matrix \( \rho \) has elements

\[
\rho_{ij} = \begin{cases} \frac{1}{1 - \sin[(m - 2j) \beta(n)/2]} + \frac{2 \sin[(m \beta(n)/2)]}{\sin[(m + 2j) \beta(n)/2]}, & \text{for } m = 0 \\ \frac{1}{\sin[(m + 2j) \beta(n)/2]}, & \text{for } m \text{ odd} \\ \frac{1}{\sin[(m - 2j) \beta(n)/2]}, & \text{for } m = 2 \\ 0, & \text{otherwise} \end{cases}
\]

where \( m = |i - j| \) and \( \beta(n) = 4 \pi \hat{f}_n^2 / f_s \).

3 SCFD Estimators with Frequency Independent Dispersion

Consider SCFD estimators which use a \( \cos \) smoothing function \( h \) which has \( h(n) = 0 \) for all odd \( n \). Then all of the frequency dependent terms in (9) are multiplied by 0 and the dispersion \( \sigma_c \) is frequency independent. Such estimators only require phase estimates from every second sample of the original analytic signal using a finite differencing operation between adjacent samples. Since the even (or odd) samples of analytic noise are uncorrelated [5], one can say that \( \sigma_c \) is frequency independent because the resampled noise sequence is white.

4 Optimal IF Estimation

4.1 Cramer-Rao Lower Bound

An optimal (efficient) estimator of a quantity is unbiased, maximum likelihood, normally distributed, and possesses the lowest possible variance for a given data set. In the case of real signals, this minimum variance is given by the following Cramer-Rao lower bound from [5].

Theorem 2 Cramer-Rao (CR) Lower Bound for a Real Signal: Let \( \hat{x}(n) = x(n) + n(n) \) where \( x(n) \) is a real sinusoid of the form \( x(n) = a_0 \cos(2 \pi f_0/n) \) and \( n(n) \) is a zero-mean white Gaussian noise sequence with variance \( \sigma_n^2 \). Then the CR lower bound on the variance of an estimator of \( f_0 \) (using only the odd or even samples of the analytic signal) is given by

\[
\text{var}[\hat{f}_0] \geq \frac{\sigma_n^2}{\sigma_0^2} \frac{\sigma_n^2}{\sigma_0^2} \frac{1}{N_s(N_s^2 - 1)}
\]

(12)

where \( N_s = (M + 1)/2 \) and \( M \) is the number of samples in the data window, which is presumed odd. The signal-to-noise ratio is given by \( s = \sigma_0^2/\sigma_n^2 \).

The length \( M \) represents the length of the periodogram window in the case of the maximum likelihood estimator (see subsection 4.3). For a SCFD estimator with a smoothing window of odd length \( P \) we use \( P = M + 2 \).

4.2 Circular Sample Estimators

We must also account for the circular nature of discrete-time frequency estimates in our definition of mean square error (MSE). We use the linearized circular mean square error (CMSE) from [5].

Definition 6 Circular Mean Square Error (CMSE): Let \( \{\hat{f}(k)\} \) be a set of \( K \) discrete-time frequency estimates of a sinusoid with frequency \( f_0 \). In noise sampled at \( f_s \) Hz. Then the sample estimator of the (linearized) CMSE is defined by

\[
\sigma_p^2 = \frac{f_s^2}{8 \pi^2} \text{ln} \left[ \frac{\text{cos}[\pi(f - f_0)/f_s]}{\sum_{k = 0}^{K-1} e^{i k f_s/k_s}} \right]
\]

(13)

where \( f_0 \) is the circular sample mean given by

\[
\hat{f}_0 = \frac{1}{K} \left( \sum_{k = 0}^{K-1} e^{i k f_s/k_s} \right)
\]

(14)

4.3 Maximum Likelihood Estimation

The optimal maximum likelihood (ML) estimator of a complex sinusoid in complex white Gaussian noise is given by the location of the peak of the periodogram [6]. A coarse estimate can be made directly from the DFT of the signal. For most frequency estimators, there is a SNR value, called the threshold, below which the CMSE of the estimator rises very rapidly as SNR decreases. This threshold generally decreases with increasing data window length \( M \) [6]. Although several estimators are optimal at high SNR, they all exhibit higher thresholds than the ML estimator for a given data window length.
4.4 The Kay Estimator

Kay [3] recently proposed an estimator based on phase differencing of the even or odd samples of the analytic signal which involves far less computation than the ML estimator and can be easily formulated as a recursion in time for real-time IF tracking. This estimator is based upon linear regression on the phase as proposed by Tretter [7]. Kay claims that his estimator is unbiased and that its variance attains the CR lower bound for moderate SNR (albeit at a somewhat higher threshold than the ML estimator). We will see that these claims are only true for certain sections of the frequency range because the Kay estimator does not handle the circular nature of discrete-time frequency estimators correctly.

We have reformulated the Kay estimator using our notational conventions so that it resembles an SCFD estimator. Kay’s estimator is given by

$$j_l^*(n) = \frac{j_s}{4\pi} \sum_{p=-Q}^{Q} h_p(p) \arg[f(n-p+1)z^*[n-p-1]]$$

$$= \sum_{p=-Q}^{Q} h_p(p)j_l(n-p)$$

where

$$h_p(p) = \begin{cases} \frac{3N_i}{2(N_i - 1)} & \text{for even } |p| \leq Q \\ 0 & \text{otherwise} \end{cases}$$

is the comb smoothing window with length \( P = 2Q + 1 \) containing \( N_i = (P+3)/2 \) independent samples (as in definition 2).

Kay uses a parabolic window function \( h_p \) which minimizes the variance of the estimator according to his analysis (which is slightly suspect since it applies linear regression to circular data). The parabolic shape arises because of the dependency between successive CFD estimates. Kay has compared the dispersion of this estimator with a rectangularly windowed estimator, which he calls the unwindowed estimator, given by

$$j_l = \frac{j_s}{4\pi} \sum_{p=-Q}^{Q} h_r(p) \arg[z(n-p+1)z^*[n-p-1]]$$

where

$$h_r(n) = \begin{cases} \frac{N_i}{N_i - 1} & \text{for even } n \leq Q \\ 0 & \text{otherwise} \end{cases}$$

He has shown that the ratio of variances of the unwindowed estimator to the parabolically windowed Kay estimator approaches \( N_i/6 \) for large \( N_i \).

Contrary to Kay’s claims we find that the Kay estimator is biased and exhibits a high threshold when the IF approaches 1 or \( f_s/2 \) Hz. This effect occurs at low frequencies, for example, because some of the \( j_l^* \) in the summation in (16) wrap around the circular domain so that estimates that should be near 0 Hz sometimes appear near \( f_s/2 \); this effect greatly increases the variance of the sum. The poor performance of the Kay estimator is a direct result of using the linear convolution operation on circular data.

1In [3] estimator bias was minimized near 0 Hz because the phases were expressed in \([-\pi, \pi]\) instead of \([0, 2\pi]\).

4.5 The PSCF D Estimator

These problems may be overcome by replacing the linear convolution operation in (16) with the modulo convolution operation from definition 4 to obtain a new estimator which we call the parabolic SCFD (PSCFD) estimator given by

$$j_l^*(n) = h_p(p) \left( \sum_{p=-Q}^{Q} h_p(p)z^*[n-p+1]z(n-p-1) \right)$$

where

$$\sigma^2 = \frac{j_s^2}{(4\pi)^2} \frac{6}{N_i(N_i^2 - 1)}$$

Now (22) is just the CR lower bound given by definition 2. It follows that \( \sigma^2 \) is the minimum value that the SCFD dispersion \( \sigma^2 \) from (9) can attain using comb smoothing window functions. When \( N_i = 2 \) the PSCFD estimator becomes the CFD estimator and so we see that the CFD estimator is also optimal. Kay’s unwindowed estimator corresponds to an SCFD estimator with a rectangular comb smoothing window which has a dispersion of

$$\sigma^2 = \frac{j_s^2}{(4\pi)^2} \frac{1}{s(N_i - 1)^2}$$

obtained by substituting (19) into (9): Dividing (23) by (22) we obtain

$$\frac{\sigma^2}{\sigma_s^2} = \frac{N_i(N_i + 1)}{6(N_i - 1)} \approx \frac{N_i}{6}$$

a result which is identical to Kay’s. We see that parabolic windowing offers substantial dispersion improvement for large window lengths.

5 Comparison of Simulation Results

Figures 1 and 2 show the performance of the ML, PSCFD and Kay estimator for normalized frequencies of \( f_s/f_s = 0.05 \) and 0.20 respectively. These figures show that the Kay estimator has a frequency dependent threshold. This estimator is also biased towards a normalized frequency of 0.5 at low SNR. This large bias causes the anomalous behaviour of the MSE of the Kay estimator for SNRs below 2 dB in Figure 1. Both the PSCFD and ML estimators are unbiased and have a constant threshold.

6 TFR Moment IF Estimators Revisited

Kay mentions two additional estimators which are equivalent to the Kay estimator (16) and the unwindowed estimator (18) at high SNR. These are respectively

$$j_l = \frac{j_s}{4\pi} \arg \left( \sum_{p=-Q}^{Q} h_p(p)z^*[n-p+1]z(n-p-1) \right)$$

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and

\[ f_{pl} = \frac{1}{4\pi} \arg \left( \sum_{p=-Q}^{Q} h_r(p)z(n-p+1)z^*(n-p-1) \right) \quad (26) \]

These estimators are formed by interchanging the operations of argument and summation. Lank, Reed and Pollon [8] proposed the estimator \( f_{pl} \) of (26) and it was later studied by Jackson and Tufts [9] and then Kay [10]. The variance of (26) is given in [8] and is identical to our result (23) at high SNR although Kay shows that \( f_{pl} \) has a higher threshold than \( f_{tr} \). The relationships in [2] show that (26) is identical to a TFR moment estimator; in fact if \( h_r \) were constant for all samples within the window rather than just the even samples, (26) would be identical to the TFR IF moment estimator derived from the spectrogram, which is the natural extension of the periodogram to non-stationary signals, with a rectangular window of length \( M = P + 2 \). Similarly (25) is the TFR moment IF estimator corresponding to the PSCFD. This does not exactly correspond to any well-known TFR. However (25) approximately corresponds to the TFR moment IF estimator from a spectrogram formed with a low-sidelobe window such as the Hamming window which is very similar to the parabolic window.

7 Conclusions

The Kay estimator was found to be biased and exhibit a large threshold for certain frequencies because it does not account for the circular nature of discrete-time frequency estimates. We replaced the linear convolution operation in the Kay estimator with the appropriate convolution operation for circular data to arrive at the PSCFD estimator. This estimator is unbiased and has frequency independent variance but it retains the optimal performance and simplicity of the Kay estimator.

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8 References


